MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

VOL. LXIX. DECEMBER 11, 1908. NO. 2

H. F. NEWALL, Esq., M.A., F.R.S., PRESIDENT, in the Chair.

The Rev. Thomas Nicklin, M.A., Bescott, Rossall Beach, Fleetwood, Lancashire,

was balloted for and duly elected a Fellow of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Samuel Ernest Loxton, Solicitor, Cannock, Stafford (proposed by W. H. Robinson);

John Mastin, F.L.S., F.C.S., etc., Woodleigh House, Totley Brook, near Sheffield (proposed by G. W. Nicolls);

Philibert J. Melotte, Royal Observatory, Greenwich, S.E. (proposed by Sir W. H. M. Christie); and

Arthur James Potts, Optician, 36 Pentland Road, Edgbaston, Birmingham (proposed by W. H. Robinson).

Seventy-three presents were announced as having been received since the last meeting, including, amongst others :—

Prof. Th. Albrecht, Formeln und Hilfstafeln für Geographische Ortsbestimmung, presented by the author; F. A. Bellamy, Historical Account of the Ashmolean Natural History Society of Oxfordshire, presented by the author; Coordinadas geograficas en la zona de totalidad del Eclipse de Sol, Aug. 1905, presented by Fr. Cortie; Sir G. H. Darwin, Collected Scientific Papers, vol. 2, Tidal Friction and Cosmogony, presented by the author; Prof. J.

Larmor, Aether and Matter (Adams Prize Essay), presented by Mr. Newall; International Union for Solar Research, Transactions, vol. 2, presented by the publishers; John Tebbutt, Astronomical Memoirs, presented by the author; Photographs of Comet *c* 1908, further series of 15 transparencies, presented by the Astronomer Royal; Photographs of the Moon, etc., taken with a cœlostat reflector, presented by Mr. J. H. Reynolds.

The Relations between Position Angle and Distance and Standard (photographic) Coordinates. By H. C. Plummer, M.A.

1. There is reason to believe that some micrometrical observers recognise some advantage in comparing photographic results with their own differential measures, but are deterred partly by a supposed difficulty in reducing the photographic measures as they appear, for example, in the Astrographic Catalogues, and partly by a distrust in the accuracy of photographic work. It may be well, therefore, to examine what is involved in a careful reduction, and to give a fully-worked example. The value of the photographic catalogues for the purpose has been pointed out, with a number of illustrative results, by Mr. Innes, in a review of Burnham's General Catalogue of Double Stars (*The Observatory*, xxx. p. 452).

2. Let (ξ, η), (ξ_1, η_1) be the standard coordinates of the principal star and companion respectively on a plate whose centre is at R.A. A and decl. D, the focal length being for the present taken as unity. Let the position angle and distance be p and s. It will be convenient to put

$$N^2 = 1 + \xi^2 + \eta^2; \quad N_1^2 = 1 + \xi_1^2 + \eta_1^2 \quad . \quad . \quad (1)$$

$$\Delta\xi = \xi_1 - \xi, \qquad \Delta\eta = \eta_1 - \eta \quad . \quad . \quad . \quad (2)$$

Then, since $(\xi, \eta, 1)$ and $(\xi_1, \eta_1, 1)$ are proportional to the direction cosines of lines drawn in the direction of the two stars, we have at once

$$\cos s = (\xi\xi_1 + \eta\eta_1 + 1)/NN_1 \quad . \quad . \quad . \quad (3)$$

which gives

$$\sin^2 s = \{\Delta\xi^2 + \Delta\eta^2 + (\eta\Delta\xi - \xi\Delta\eta)^2\}/N^2N_1^2 \, . \quad . \quad (4)$$

Now, on an astrographic plate, ξ and η can scarcely exceed $\frac{1}{60}$, and are usually much less. Hence, if we wish to measure distances up to 300″ and allow an error of 0″·1 in extreme cases on such plates we can put simply

$$s = (\Delta\xi^2 + \Delta\eta^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (5)$$

On larger plates it may be necessary to use a closer approximation

3. If the principal stars were at the centre of the plate, ξ_1, η_1 would give immediately p and s. But in any case we can suppose a transformation to a second plate so as to bring the principal star to the centre and the other star to the point (ξ', η'). It will be convenient to put

$$L^2 = \xi^2 + (\cos D - \eta \sin D)^2 \quad . \quad . \quad . \quad (6)$$

Then we shall have

$$\xi' = \tan s \sin p$$

$$= \{(\cos D - \eta \sin D)\Delta\xi + \xi \sin D.\, \Delta\eta\} N/(\xi\xi_1 + \eta\eta_1 + 1)L \quad . \quad (7)$$

$$\eta' = \tan s \cos p$$

$$= \{ -\xi(\sin D + \eta \cos D)\Delta\xi + [(1 + \xi^2)\cos D - \eta \sin D]\Delta\eta\}$$
$$/(\xi\xi_1 + \eta\eta_1 + 1)L \quad (8)$$

Exact transformation formulæ in this precise form, involving only ξ, η and D, do not seem to have been given before, and proofs will be found at the end of this paper. Hence, neglecting terms multiplied by ξ^2, $\xi\eta$ and η^2,

$$\tan p = \frac{(\cos D - \eta \sin D)\Delta\xi + \xi \sin D.\Delta\eta}{-\xi \sin D.\Delta\xi + (\cos D - \eta \sin D)\Delta\eta}$$

or

$$p = \tan^{-1}\Delta\xi/\Delta\eta + \tan^{-1}\{\xi \tan D/(1 - \eta \tan D)\} \quad . \quad (9)$$

The terms neglected in (7) and (8) do not increase with D, and therefore (9) holds for small plates in any part of the sky. But, except in the immediate neighbourhood of the pole, a further simplification can be made, and we may write

$$p = \tan^{-1}\Delta\xi/\Delta\eta + \xi \tan D \quad . \quad . \quad . \quad . \quad . \quad (10)$$

4. Thus far it has been assumed that the standard coordinates are referred to unit focal length, *i.e.* are expressed, so to speak, in circular measure. But it is clear that in applying the simple formulæ (5) and (10) this restriction is no longer necessary, and that the standard coordinates may be taken in any definite angular unit. Moreover, the catalogues give measured, not standard coordinates. The total want of uniformity in the presentation of their results by the different observatories is regrettable, and the following remarks will strictly apply only to the Oxford Catalogue. We have

$$\xi = x - Ax - By - C - 13$$
$$\eta = y - Dx - Ey - F - 13$$

the plate-constants being given with the measured coordinates x, y. Hence

$$\Delta\xi = \Delta x - A.\Delta x - B.\Delta y, \quad \Delta\eta = \Delta y - D.\Delta x - E.\Delta y \quad . \quad (11)$$

where $\Delta x = x_1 - x$ and $\Delta y = y_1 - y$ are expressed in the unit 5′.

The plate-constants are small, so that approximately $\xi = x - 13$ and the equations (5) and (10) may be re-written in the form

$$s = (\Delta\xi^2 + \Delta\eta^2)^{\frac{1}{2}} \cdot 300'' \quad . \quad . \quad . \quad (12)$$

$$p = \tan^{-1}\Delta\xi/\Delta\eta + 1^\circ.(x - 13)\tan D/12 \quad . \quad (13)$$

s and p being thus expressed, as usual, in seconds and degrees respectively.

5. As an illustration, chosen at random, I have taken the case of the stars in the neighbourhood of υ Coronæ, which is 7566 in the Cambridge A.G. Catalogue, and 7570 in Burnham's General Catalogue. The stars required are easily identified on the plates of two different zones, and the following extracts are set down :—

R.A. $16^h\ 16^m + 29^\circ$.

Plate 1011 ; 1896 April 21.

A.	B.	C.
+·00772	+·00834	−·3619
D.	E.	F.
−·00807	+·00760	−·1459

Mag. $= 14{\cdot}1 - 1{\cdot}02\sqrt{d}$.

No.	d.	x.	y.
38718*	75	4·304	17·754
719	10	4·385	17·989
729	18	4·416	18·020
730	14	4·634	18·005

R.A. $16^h\ 12^m + 30^\circ$.

Plate 1952 ; 1902 May 27.

A.	B.	C.
−·00084	+·00024	−·0855
D.	E.	F.
−·00002	−·00064	−·3676

Mag. $= 16{\cdot}0 - 1{\cdot}25\sqrt{d}$.

No.	d.	x.	y.	Star.
35790*	80	14·843	5·398	A
791	13	14·920	5·635	E
792	23	14·952	5·666	C
793	22	15·167	5·653	D

	AC.	AD.	CE.	AC.	AD.	CE.
Δx	+·1120	+·3300	−·0310	+·1090	+·3240	−·0320
− A.Δx	− 9	− 25	+ 2	+ 1	+ 3	0
− B.Δy	− 22	− 21	+ 3	− 1	− 1	0
$\Delta\xi$	+·1089	+·3254	−·0305	+·1090	+·3242	−·0320
Δy	+·2660	+·2510	−·0310	+·2680	+·2550	−·0310
− D.Δx	+ 9	+ 27	− 3	0	0	0
− E.Δy	− 20	− 19	+ 2	+ 2	+ 2	0
$\Delta\eta$	+·2649	+·2518	−·0311	+·2682	+·2552	−·0310
$\sqrt{(\Delta\xi^2 + \Delta\eta^2)}$	·2864	·4114	·0436	·2895	·4126	·0446
s	85″·92	123″·42	13″·08	86″·85	123″·78	13″·38
$\tan^{-1}\Delta\xi/\Delta\eta$	22°·4	52°·3	224°·4	22°·1	51°·8	225°·9
$(x-13)/12$	−0·72	−0·72	−0·72	+ 0·15	+ 0·15	+0·16
× tan D	−0°·4	−0°·4	−0°·4	+0°·1	+0°·1	+0°·1
p	22°·0	51°·9	224°·0	22°·2	51°·9	226°·0

6. These results, which are referred to the equinox 1900·0, can now be compared with the micrometer observations given in Burnham's General Catalogue.

υ Coronæ = Burnham 7570.

p.	*s.*	Mags.	Ep.	Obs.	
		A and B.			
29°·5	55″·98	3	12·0	1879·32	β 2
28·7	55·45		12·7	1898·01	Doo. 2n
28·7	54·50		13	1905·05	β 1n

There is no image of the star B on either of the Oxford plates. The photographic magnitude is presumably fainter than 12·5.

		A and C.			
24·5	88·69		(13)	1823·36	Sh 2
22·4	86·31		9·3	1879·32	β 2n
22·0	85·92	5·3	9·8	1896·30	Oxf.
22·4	85·35		8·5	1897·90	Doo. 5n
22·2	86·85	4·8	10·0	1902·40	Oxf.
22·1	86·67		10·8	1905·05	β 1n
		A and D.			
54·9	126·42		(12)	1823·36	Sh. 2
52·2	123·61		9·0	1879·32	β 2n
51·9	123·42	5·3	10·3	1896·30	Oxf.
51·9	123·70		8·0	1897·90	Doo. 5n
51·9	123·78	4·8	10·1	1902·40	Oxf.
51·5	123·75		10	1905·05	β 1n
		C and E.			
222·7	13·23	9·3	10·5	1879·32	β 2
224·0	13·08	9·8	10·9	1896·30	Oxf.
226·0	13·38	10·0	11·5	1902·40	Oxf.
226·0	13·34	10·8	11·5	1905·05	β 1n

7. The evidence given by this group of four stars occurring on two different plates suggests the following inferences:—

(1) Differential measures of pairs of stars can be obtained from the published Astrographic Catalogues with very little trouble.

(2) Such measures have an accuracy of the same class as the results of the very best visual observers.

(3) The photographic magnitudes are, at any rate, not inferior to the estimates of the skilled double-star observer.

(4) If the foregoing conclusions are justified, some guidance may be obtained as regards the policy which the observer at the

telescope should follow. In general, the photographic method does not allow of the accurate measurement of very close companions; the limit of separation required depends on the magnitude of the components. Also the faintest stars lie outside the great mass of easily accessible data which the Astrographic Catalogues will contain. There is thus an ample field in which the work of the visual observer will be complementary to the photographic results. But in future care ought to be exercised not to trench on ground where visual observations will be purely redundant.

8. Proofs will now be given of the transformation formulæ (7) and (8) which were employed in § 3. Let (x, y, z) be the coordinates of a point referred to the rectangular axes OA, OB, OC, where C is the plate centre, R.A. = A and decl. = D, and the plane OCB passes through the pole P. Let also (x', y', z') be the coordinates of the same point referred to the axes OA′, OB′, OC′, where C′ is a new plate centre, R.A. = α and decl. = δ, and the plane OC′ B′ passes through the pole P. The relations between the two systems will be given by a transformation scheme

	x'	y'	z'
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

Now if (x'', y'', z'') be the coordinates of the same point referred to the axes Oγ, OL, OP (γ being the first point of Aries), the relations of these to (x, y, z) are given by

	x''	y''	z''
x	− sin A	cos A	0
y	− sin D cos A	− sin D sin A	cos D
z	cos D cos A	cos D sin A	sin D

with a similar scheme for x', y', z', in terms of α and δ. Hence, when x'', y'', z'' are eliminated, we find

$$
\begin{aligned}
l_1 &= \cos(\alpha - A)\\
m_1 &= -\sin\delta\sin(\alpha - A)\\
n_1 &= \cos\delta\sin(\alpha - A)\\
l_2 &= \sin D\sin(\alpha - A)\\
m_2 &= \sin\delta\sin D\cos(\alpha - A) + \cos\delta\cos D\\
n_2 &= -\cos\delta\sin D\cos(\alpha - A) + \sin\delta\cos D\\
l_3 &= -\cos D\sin(\alpha - A)\\
m_3 &= -\sin\delta\cos D\cos(\alpha - A) + \cos\delta\sin D\\
n_3 &= \cos\delta\cos D\cos(\alpha - A) + \sin\delta\sin D
\end{aligned}
$$

Now ξ, η being the coordinates of C′ on the plate with centre C,

$$
\begin{aligned}
N\cos\delta\sin(\alpha - A) &= \xi\\
N\sin\delta &= \sin D + \eta\cos D\\
N\cos\delta\cos(\alpha - A) &= \cos D - \eta\sin D
\end{aligned}
$$

so that

$$N\cos\delta = L,$$

where N and L have the meanings defined by (1) and (6). Hence we easily derive

$$\begin{aligned}
l_1 &= (\cos D - \eta \sin D)/L \\
m_1 &= -\xi(\sin D + \eta \cos D)/LN \\
n_1 &= \xi/N \\
l_2 &= \xi \sin D/L \\
m_2 &= \{(1+\xi^2)\cos D - \eta \sin D\}/LN \\
n_2 &= \eta/N \\
l_3 &= -\xi \cos D/L \\
m_3 &= \{(\xi^2+\eta^2)\sin D - \eta \cos D\}/LN \\
n_3 &= 1/N
\end{aligned}$$

But since ξ', η' and ξ_1, η_1 are corresponding coordinates on the plates whose centres are C′ and C respectively,

$$\begin{aligned}
\xi' &= x'/z' = (l_1\xi_1 + l_2\eta_1 + l_3)/(n_1\xi_1 + n_2\eta_1 + n_3) \\
\eta' &= y'/z' = (m_1\xi_1 + m_2\eta_1 + m_3)/(n_1\xi_1 + n_2\eta_1 + n_3)
\end{aligned}$$

and therefore

$$\begin{aligned}
\xi' &= \{(\cos D - \eta \sin D)\,\xi_1 + \xi \sin D.\eta_1 - \xi \cos D\}N/(\xi\xi_1 + \eta\eta_1 + 1)L \\
\eta' &= [-\xi(\sin D + \eta \cos D)\xi_1 + \{(1+\xi^2)\cos D - \eta \sin D\}\,\eta_1 + \\
&\qquad (\xi^2+\eta^2)\sin D - \eta \cos D]/(\xi\xi_1 + \eta\eta_1 + 1)L
\end{aligned}$$

which by (2) become identical with (7) and (8).

9. Similar expressions may be obtained in another way. Let S_0 be the principal star and S_1 the second star. With respect to the plate centre C let p_0, s_0 be the position-angle and distance of S_0, p_1 and s_1 of S_1, so that

$$\xi = \tan s_0 \sin p_0\,,\quad \eta = \tan s_0 \cos p_0\,,\quad \cos s_0 = 1/N$$

$$\xi_1 = \tan s_1 \sin p_1\,,\quad \eta_1 = \tan s_1 \cos p_1\,,\quad \cos s_1 = 1/N_1$$

Then

$$\begin{aligned}
\sin PS_0 \sin s \sin p &= \sin PS_0 \sin PS_1 \sin (CPS_1 - CPS_0) \\
&= \sin s_1 \sin p_1 (\cos D \cos s_0 - \sin D \sin s_0 \cos p_0) \\
&\quad - \sin s_0 \sin p_0 (\cos D \cos s_1 - \sin D \sin s_1 \cos p_1) \\
&= \{\xi_1(\cos D - \eta \sin D) - \xi(\cos D - \eta_1 \sin D)\}/NN_1 \\
\sin PS_0 \sin s \cos p &= \cos PS_1 - \cos S_0P \cos S_0S_1 \\
&= \sin D \cos s_1 + \cos D \sin s_1 \cos p_1 \\
&\quad - (\sin D \cos s_0 + \cos D \sin s_0 \cos p_0). \\
&\quad \{\cos s_1 \cos s_0 + \sin s_1 \sin s_0 \cos (p_0 - p_1)\} \\
&= (\sin D + \eta_1 \cos D)/N_1 \\
&\quad - (\sin D + \eta \cos D)(1 + \xi\xi_1 + \eta\eta_1)/N^2N_1 \\
&= [\sin D\,(\xi^2 + \eta^2 - \xi\xi_1 - \eta\eta_1) \\
&\quad + \cos D\{\eta_1 - \eta + \xi(\xi\eta_1 - \xi_1\eta)\}]/N^2N_1 \\
\sin^2 PS_0 &= 1 - (\sin D \cos s_0 + \cos D \sin s_0 \cos p_0)^2 \\
&= 1 - (\sin D + \eta \cos D)^2/N^2 = L^2/N^2
\end{aligned}$$

Hence we have

$$\sin s \sin p = \{\cos \mathrm{D}.\Delta\xi - \sin \mathrm{D}(\eta\Delta\xi - \xi\Delta\eta)\}/\mathrm{LN}_1$$

$$\sin s \cos p = [-\sin \mathrm{D}(\xi\Delta\xi + \eta\Delta\eta) + \cos \mathrm{D}\{\Delta\eta - \xi(\eta\Delta\xi - \xi\Delta\eta)\}]/\mathrm{LNN}_1$$

which, when (3) is remembered, are easily seen to be equivalent to (7) and (8).

University Observatory, Oxford:
1908 *December* 10.

Analysis of the Colours and Magnitudes of 3630 *Stars between the N. Pole and* 25° *S. Declination.* By W. S. Franks.

During the recent investigation on star colours and spectra (*Monthly Notices*, lxviii. 672), it occurred to me that an analysis of the colours and magnitudes might form a fitting sequel, and might possibly furnish some additional information. I therefore tabulated each hour of R.A. separately into 6 magnitude groups and 7 colour groups (as before); the magnitudes were all based on the *Revised Harvard Photometry*, and the minimum limit was taken at 6·5 of that scale. Of course this list does not contain *all* the 6·5 magnitudes to be found in the *R.H.P.*, but only those observed. The *comites* of double stars were carefully excluded. It is unnecessary waste of space to give the extended details here; suffice it to say that they were all ultimately arranged in four divisions—two galactic (hours III to VIII, and XV to XX); and two non-galactic (hours XXI to II, and IX to XIV). These were again combined into two—galactic and non-galactic—as here shown:—

A. *Combined galactic regions.*

Magnitude.	White.	Yellowish-White.	Pale Yellow.	Yellow.	Pale Orange.	Orange.	Orange-Red.	Total.
>1·5	4	3	1	1	...	1	1	11
1·6–2·5	14	1	1	1	...	...	...	17
2·6–3·5	24	9	15	19	...	2	...	69
3·6–4·5	90	42	45	42	9	4	...	232
4·6–5·5	268	166	139	84	48	27	1	733
5·6–6·5	292	168	208	82	82	51	10	893
Total	692	389	409	229	139	85	12	1955

B. *Combined non-galactic regions.*

Magnitude.	White.	Yellowish White.	Pale Yellow.	Yellow.	Pale Orange.	Orange.	Orange-Red.	Total.
>1·5	2	...	1	1	...	...	...	4
1·6–2·5	8	1	2	9	...	1	...	21
2·6–3·5	10	10	8	11	2	3	...	44
3·6–4·5	43	32	39	31	5	6	1	157
4·6–5·5	159	107	126	72	44	29	5	542
5·6–6·5	221	164	231	115	97	73	6	907
Total	443	314	407	239	148	112	12	1675

C. *The whole together.*

Magnitude.	White.	Yellowish White.	Pale Yellow.	Yellow.	Pale Orange.	Orange.	Orange-Red.	Total.
>1·5	6	3	2	2	...	1	1	15
1·6–2·5	22	2	3	10	...	1	...	38
2·6–3·5	34	19	23	30	2	5	...	113
3·6–4·5	133	74	84	73	14	10	1	389
4·6–5·5	427	273	265	156	92	56	6	1275
5·6–6·5	513	332	439	197	179	124	16	1800
Total	1135	703	816	468	287	197	24	3630

It will be at once seen that the relative distribution of colours is not the same in A as B; but that there is a great excess of white stars in the galaxy regions, and a decided preponderance of coloured stars in the non-galactic regions. This is better shown by the corresponding percentages:—

	A.	B.	C.
White	35·4	26·4	31·2
Yellowish-white	19·9	18·8	19·4
Pale yellow	20·9	24·3	22·5
Yellow	11·7	14·2	12·9
Pale orange	7·1	8·8	7·9
Orange	4·4	6·7	5·4
Orange-red	0·6	0·8	0·7
	100·0	100·0	100·0

There is thus an undoubted physical connection between the colour of the stars and the galaxy; and, in the light of spectroscopic evidence, it looks as if the galaxy was the newest and most vigorous part of the stellar universe, since nearly all the active sidereal phenomena are only to be found in that region. When we consider its association with Helium stars, "Wolf-Rayet" stars, bright-line spectra, short-period variables, *novæ*, etc., we are driven to the conclusion that here is the storehouse of energy for the sidereal universe, and such enormous potentiality cannot belong to any but the youngest and most active stage of existence.

Uxbridge:
1908 *December* 1.

Note on Spectral Class and Stellar Colours. By Julia Bell, M.A.

(Communicated by Professor Karl Pearson, F.R.S.)

In the *Monthly Notices*, vol. lxvii. pp. 539–42, Mr. W. S. Franks published what is statistically termed a contingency table, showing the dependence of colour on spectral class for 928 northern stars from -25° to N. Pole. The statistical reduction of this table was made by Dr. Alice Lee, and published in a paper on the "Correlation of Stellar Characters," *Monthly Notices*, lxviii. p. 422, by Miss W. Gibson and Professor K. Pearson. The coefficient of mean square contingency was found to be $C_1 = \cdot 74$, thus showing a very considerable measure of relationship between stellar colour and spectral class.

In the last issued *M.N.* (vol. lxviii. p. 673) Mr. Franks provides a further contingency table for northern stars between the same limits of declination, but including nearly four times as many stars, namely 3497. He rightly insists on the close association of the two characters, spectral class and colour class. It will not, I think, be without interest to consider what is the exact numerical measure of this relationship, and whether it is in keeping with the earlier value, as found from a much smaller sample.

Contingency Table for Northern Stars, -25° to N.P.

Colour Class.

Spectral type.	White.	Yellowish-White.	Pale Yellow.	Yellow.	Pale Orange.	Orange.	Orange-Red.	To
Bright line Spectra	0 (− ·62)	0 (− ·42)	2 (+ 1·55)	0 (− ·24)	0 (− ·15)	0 (− ·11)	0 (− ·01)	
Intermediate	5 (+ 1·90)	4 (+ 1·92)	1 (− 1·24)	0 (− 1·18)	0 (− ·78)	0 (− ·56)	0 (− ·07)	
Orion Type	191 (+116·06)	50 (− ·45)	1 (− 53·11)	0 (− 28·58)	0 (− 18·75)	0 (− 13·49)	0 (− 1·66)	2
Sirius Type	793 (+424·4)	338 (+ 89·93)	59 (−207·11)	0 (−140·54)	0 (− 92·22)	0 (− 66·36)	0 (− 8·17)	11
α Carinæ Type	78 (+ 56·41)	204 (+113·52)	143 (+ 45·95)	9 (− 42·26)	0 (− 33·63)	0 (− 24·20)	0 (− 2·98)	4
Procyon Type	7 (− 27·37)	47 (+ 23·86)	51 (+ 26·18)	4 (− 9·11)	2 (− 6·60)	0 (− 6·19)	0 (− ·76)	1
Capella Type	9 (−128·51)	69 (− 23·56)	242 (+142·71)	68 (+ 15·56)	41 (+ 6·59)	13 (− 11·76)	2 (− 1·05)	4
Arcturus Type	0 (−245·90)	17 (−148·52)	260 (+ 82·45)	274 (+180·23)	155 (+ 93·47)	84 (+ 39·73)	4 (− 1·45)	7
Aldebaran Type	0 (− 33·76)	0 (− 22·72)	12 (− 12·37)	32 (+ 19·13)	29 (+ 20·55)	36 (+ 29·92)	0 (− ·75)	
Betelgeuse Type	0 (− 46·45)	0 (− 31·27)	11 (− 22·54)	26 (+ 8·29)	44 (+ 32·38)	59 (+ 50·64)	10 (+ 8·97)	
19 Piscium Type	0 (− 3·41)	0 (− 2·30)	0 (− 2·46)	0 (− 1·30)	0 (− ·85)	3 (+ 2·39)	8 (+ 7·92)	
Total	1083	729	782	413	271	195	24	3

The accompanying table shows Mr. Franks' data, providing by the numbers in brackets the deviations in the observed frequencies from the frequencies which would occur if the two characters were independent. The value of the coefficient of mean square contingency is

$$C_1 = \cdot 756,$$

with a probable error between ·01 and ·02.

Mr. Franks' much larger sample thus entirely confirms the degree of association found from the earlier data, and may serve to give confidence to those who doubt whether the numbers found to express association of characters by modern statistical methods are really characteristic of stellar populations. The wide change in the constitution of the sample has not modified the intensity of dependence within the limits of the probable error of sampling.

One further point: the association here found is essentially one of *qualitative* classes, and it is difficult to imagine a single *quantitative* unit at the basis of either character, *i.e.* we are dealing with a case where contingency and not correlation methods must be adopted. (*M.N.*, vol. lxviii. p. 417) If, for a moment, we could suppose two *correlated* variables following the Gaussian law of frequency behind both colour class and spectral type, we should be rudely awakened to the absurdity of the hypothesis by noting that the coefficient of mean contingency* for this table is

$$C_2 = \cdot 86,$$

and the equality $C_1 = C_2$ necessitated by the above hypothesis is wholly unfulfilled.

The purpose of this note is to emphasise two points:—

(1) The stability of statistical constants when drawn from different samples of the stellar population.

(2) The need for remembering, in applying modern statistical methods to astronomical problems, that correlation is only a single and not very long chapter in the complete theory of association: the chapter on contingency may often be of greater service.

* "On the Theory of Contingency and its relation to Association and Normal Correlation," by K. Pearson, *Drapers' Research Memoirs*, Dulau & Co., 1904.

University College, London:
1908 *November* 14.

Note on some Photographic Images of Mars taken in 1907 *by Professor Lowell.* By E. M. Antoniadi. (Plate 4.)

Some time ago Professor Percival Lowell, Director of the Flagstaff Observatory in Arizona (U.S.A.), had kindly sent to Mr. A. C. D. Crommelin a print showing 40 splendid images of Mars, taken by him on 1907 July 11, with various exposures, the mean longitude of the centre of the disk being nearly 250° at the time.

This print was handed on to me by Mr. Crommelin.

The amount of detail shown on these photographs is very considerable, and I have deemed it worth the while to submit these images to a thorough examination, and to group the various features shown on a large scale drawing. I have also decided to show no marking, traces of which could not be found on more than one image.

The fair publication of such a drawing is by no means an easy matter,—first, owing to the diffuse limits of the dusky spots, and to the very delicate gradations of the shading rendering interpretation exceedingly difficult; and secondly, owing to obstacles attending reproduction. The latter, however, can be avoided to a large extent by distrusting continuous grey shading, whose reproduction necessitates retouching on the part of the photographic engraver, and by boldly decomposing the duskiness into very small black dots and lines on a white grouud. Upon a careful comparison with the originals, fig. 1, Plate 4, will be found satisfactory, as following closely the general appearance of the disk, except in the diffuseness of the limb and in the protrusion of white spots in its vicinity.

My observations of these photographs may be given as follows:—

(*a*) *Varying sharpness and appearance of the markings.*—Owing to atmospheric tremors, the 40 images differ widely in the amount of detail recorded. In addition to this, the self-same marking presents itself under varying forms. The two lakes at each end of the canal called *Cerberus*, for instance, are quite large, dark, and distinct on some images, while on others they are quite indistinguishable from the *Cerberus* mass. What at times appeared as a perfect dumb-bell was a few moments later distorted into a streak. This is a very instructive and *à priori* incredible fact, and one, too, which accounts for the greatly differing representations of the same object by different observers, or for the variety of sensations experienced by single observers on a given night.

(*b*) *The albedo of the limb.*—Another most interesting fact revealed by the photographs is that the superior brightness of the limb, which is so striking a feature in visual observations, especially under low powers, is entirely non-existent. The phenomenon is thus proved to be due to fatigue of the eye and to contrast,* so that the atmospheric layer above the visible markings of the

* *Memoirs B.A.A.*, vol. x., part iii., p. 138.

FIG. 1.—Drawing of Mars, showing all trustworthy details visible on photographs of the planet taken by Prof. Lowell on 1907 July 11. $\omega = 250°$.

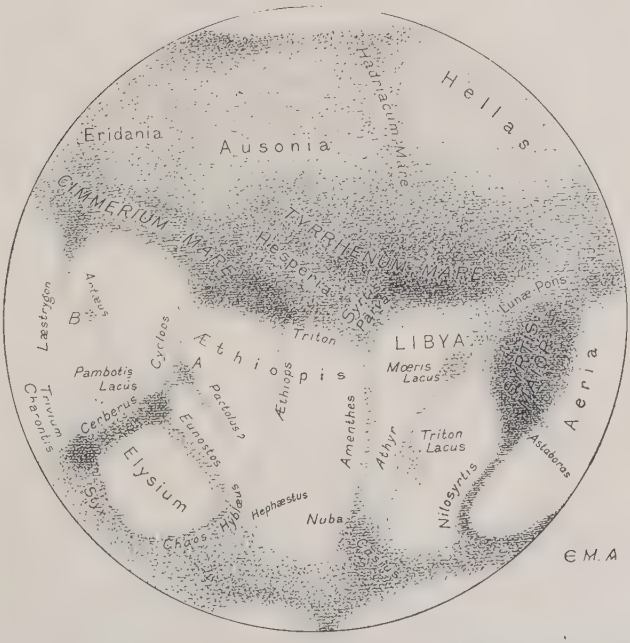


FIG. 2.—Key chart.

planet is virtually destitute of particles capable of reflecting sunlight. Hence, to an observer on Mars, as on the Moon, the stars would be readily visible on a dark sky in broad daylight.*

(*c*) *The lands whitening with the obliquity.*—It is impossible to speak authoritatively on this subject, as I have examined only the above 40 photographs, but from what is shown on them, my suspicion regarding the at least partial subjectivity of this phenomenon † seems confirmed, inasmuch as all three markings of this class visible near the limb on the photographs, namely *Hellas*, *Eridania* and *Aeria*, show no trace of increased brightness, while *Eridania* is even abnormally dark in the vicinity of the left limb.

(*d*) *The edges of the dark areas.*—As is well known, Professor Schiaparelli has depicted the *Maria* as bordered by darker canals towards the so-called continental regions whenever the markings were seen to double,‡ but no trace whatever of such a structure is discernible on the photographs.

(*e*) *The snow cap.*—The N. edge of the S. polar cap is almost straight, and not so convex to the centre of the disk as shown by visual observations. There is, moreover, a slight protuberant extension of the snow towards longitude 210°, and a very shallow recess towards longitude 245°. The dusky band round the cap is faint, except to the left, over *Thyle Insulæ*.

(*f*) *The large dusky areas.*—In accordance with Schiaparelli's map, that part of the S. hemisphere included between the N. edges of the *Maria* and the polar cap is dusky, or at least less photogenic photographically than the so-called continental regions, and we know that the islands in these expanses are of a ruddy tinge.

There is a broad dusky streak emerging from the snow cap towards longitude 275°, and trending in a slight curve into *Syrtis Minor*, although losing itself in the darkness of *Mare Tyrrhenum*. This seems to be *Hadriacum Mare* (see the key-chart, fig. 2). *Hellas* is large, and its appearance usual; *Ausonia* confused as a large mass extending as far as the polar land; *Mare Chronium* indistinct; *Eridania* dark. *Mare Tyrrhenum* is very dark above *Libya*, the darkness extending further S. than on Schiaparelli's charts.§ *Hesperia* is also very shaded, and seems to emerge rather from the faintish *Syrtis Parva* than from *Æthiopis*, which is contrary to what has been generally observed.

Mare Cimmerium is sharply pointed on *f.* side,‖ and shows two estuaries, that of *Cyclops* and that of *Læstrygon*, the latter very deeply indenting the yellow expanse to N. The darkest part of *Mare Cimmerium* was that between the *Cyclops* and the *f.* end.

Syrtis Major appears under the form given to it by Professor Lowell in 1894, and since observed at all apparitions of the planet.

* *Ibid.*, vol. ix., part iii., p. 74. † *Ibid.*, vol. ix., part iii., p. 104.
‡ *Knowledge*, 1903, pp. 247–248. § *Memorie*, i. to iv., plates.
‖ Major Molesworth has observed this in 1898–1899; also Cerulli during the same apparition (*Nuove Osservazioni di Marte*, Collurania, 1900).

Lunæ Pons is broad and diffuse. *Libya* also showed in 1907 Lowell's flattened form of 1894, rather than Schiaparelli's arched outline to S. and its S.W.* end was dusky.

(*g*) *Lakes.*—Seven of these objects are shown on the print: *Mœris Lacus* as a diffuse shading, situated on one-third of the way from the coast of *Syrtis Major* to *Amenthes*;† *Triton Lacus*, faint, broad and diffuse; *Pambotis Lacus* ‡ and *Trivium Charontis*, as two very conspicuous dark spots; *Nuba*, a diffuse shading on S.E. end of *Casius*; lastly the lake marked A on fig. 2, between *Pambotis Lacus* and *Mare Cimmerium*, and lake B, north of the *Læstrygon* estuary. Both these last objects were very difficult on the photographs.

(*h*) *Canals.*—Seventeen of these are more or less discernible on the images, corresponding, with one exception, to canals of Schiaparelli's charts. These are—

Æthiops.—Very faint and almost imperceptible traces of this canal.

Amenthes.—Broad and diffuse, fairly easy.

Antæus.—Apparently the edge of a faint duskiness S. of *Cerberus.*

Astaboras.—Visible near its estuary, very faint.

Athyr.—Very faint, broad and diffuse, running from *Casius* into *Syrtis Major* viâ *Triton* and *Mœris Laci.*

Casius.—A broad dark marking in N. hemisphere.

Cerberus.—Quite plain, dark and broad.

Chaos.—Edge of a shading around *Elysium.*

Cyclops.—Faint and diffuse, curving to E. of its estuary.

Eunostos.—Easy.

Hephæstus.—A dusky smudge, very faint and diffuse.

Hyblæus.—Faint, but distinct.

Læstrygon.—Traces near *Mare Cimmerium*; might be the E. edge of the indefinite shading N. of *Cerberus.*

Nilosyrtis.—Quite plain and dark, though less so than *Cerberus.* It curves admirably to N.W.

Pactolus.—A very easy streak, running almost parallel to *Eunostos.*

Styx.—The dark edge of *Elysium* to N.E.

Triton.—Visible only as an intensification of the N. edge of *Hesperia.*

All these canals are diffuse.

(*i*) *Duskiness and white spots of N. hemisphere.*—The whole of the regions N. of *Elysium*, *Nuba*, and *Nilosyrtis* is slightly duskier than the equatorial regions.§ *Elysium* appears whiter than any

* E. and W. are used here in their areographic sense.

† This lake has been seen thus by Mr. R. D. Givin of Sydney (N.S.W.) at the last opposition.

‡ Discovered apparently by Major Molesworth in 1896, *Memoirs B.A.A.*, vol. vi., part iii., p. 92, and chart.

§ On the objectivity of these shadings of the N. hemisphere, which are missing in some maps of Schiaparelli and Lowell, stress has been laid in *Memoirs B.A.A.*, vol. xi., part iii., p. 91. It would appear that the glare

country of the torrid zone, while two dull white spots of unequal size protrude beyond the N. limb, and are divided by a narrow dark channel.

(*k*) *Evidence of change in the form and intensity of the dusky areas.*—Schiaparelli's long series of observations have established the modifications in tint of the Martian *Maria*. In 1888, for instance, he found *Mare Acidalium*, which is, during the spring of the N. hemisphere, by far the darkest area on the planet, greatly discoloured,* and this discovery of seasonal change has been fully corroborated by the joint results of the Mars Section of the British Astronomical Association as the *Mare*, which was black in 1896–1897, appeared paler at every succeeding apparition, so that in 1907 it was by no means darker than the southern seas, according to the Australian observers. But if the alterations in tint of the

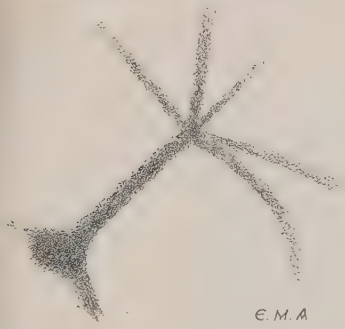


FIG. 3.—Invisibility of *Pambotis Lacus* from 1877 to 1888 (Schiaparelli).

FIG. 4.—Appearance of *Pambotis Lacus* in 1907 (Prof. Lowell's photographs).

dusky spots are unquestionable, the changes in their outline do not seem to have been hitherto more than probable. I think, however, that the valuable photographs of Professor Lowell now furnish serious evidence, and even positive proof, of such modifications. Thus while, from 1877 to 1888, Schiaparelli, who was seeing much more delicate detail than is shown on the photographs, could never detect any condensation on the junction of *Cerberus* and *Cyclops* (fig. 3), the photographs of 1907 show here a huge lake, *Pambotis Lacus*, as easy to see as *Trivium Charontis* (fig. 4). The change is absolutely unquestionable, since Schiaparelli's exhaustive scrutiny could by no means have missed a spot visible at a glance on photographic discs of Mars measuring 4 millimetres in diameter. The alterations in the E. border of *Syrtis Major* tend to the same conclusion; for while that great dusky area had always the form of a regular V to Schiaparelli (fig. 5), the photographs show it

of the Martian image in a large aperture is inimical to the visibility of these faint duskinesses. The discovery of Saturn's dark spots by Mr. Stanley Williams with a 6½-inch reflector and their invisibility to Prof. Barnard with the 36-inch Lick telescope may be accounted for in a similar manner.

* *Memoria sesta*, p. 38.

distinctly to bulge to N.E., a modification which is also beyond the limits of observational errors. And thus the reality of change in the appearance of the grey markings is established on a strong basis.

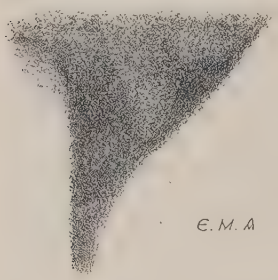


FIG. 5.—*Syrtis Major*, as it was generally seen from 1877 to 1888 (Schiaparelli).

FIG. 6.—*Syrtis Major*, after the Lowell photographs of 1907.

Professor Lowell's photographs have given a great impetus to areography by enabling us to differentiate between objective and subjective appearance much more soundly than heretofore; and I should be glad to submit to a similar analysis any further prints I may have the good fortune to receive.

1908 *September* 23.

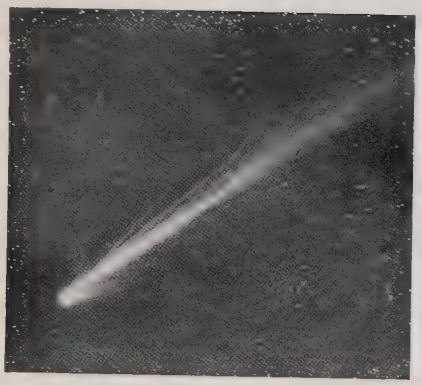
Photographs of Comet c 1908 (*Morehouse*).

By E. E. Barnard. (Plate 5.)

The following lantern slides which I have made are a continuation of those presented at the November meeting of the Society. They are from the most interesting of the negatives that I have made of the comet with the 10-inch Bruce doublet by Brashear.

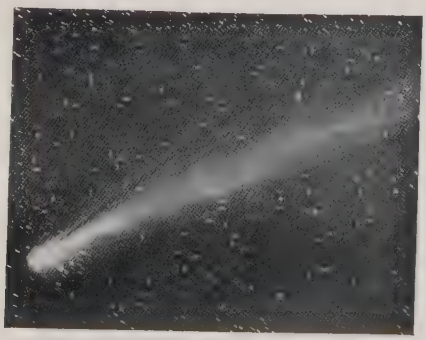
List of Lantern Slides (*continued from p.* 53).

No.	d	h	m	
15	1908 Oct. 16	9	32	Centr. St. Time $6^h\ 0^m$ slow of G.M.T.
16	29	8	20	
17	Nov. 14	6	47	
18	15	6	6	
19	16	7	10	
20	17	7	37	
21	18	7	25	
22	19	6	9	



1908 Nov. 15 d. 6 h. 6 m. C.S.T.

1908 Nov. 16 d. 7 h. 10 m. C.S.T.



1908 Nov. 17 d. 7 h. 37 m. C.S.T.

COMET c 1908 MOREHOUSE.—E. E. BARNARD.

There have been no further remarkable changes in the comet like those of September 30 and October 15. The character of the phenomena has changed, however. In these later pictures the new phenomena presented have consisted mainly in long and slender streamers and of convoluted forms in the tail itself. These have been quite characteristic of its appearance up to the beginning of the bad weather from November 21, since which time it has not been possible to see the comet. There have been variations in these features, but no regular departure from them. Every one of these later photographs is of singular interest and beauty. That of November 15 is perhaps the most remarkable. On this date there are certain deflections in the streamers which are suggestive of an outside disturbing influence of some kind. In this picture the main tail widens out from a narrow neck attached to a rather small head, from each side of which broad diffused brushes of light stream forth at considerable angles to the tail. Those on the north side extend for several degrees. On the south side of the comet two slender streamers, apparently originating in the main tail near the head, run back at a slight angle for some distance. The outer one of these, about 2° from the head, rather suddenly bends to the south, as if deflected in its course by some disturbing influence; it then seemingly returns to its original direction. A similar appearance is presented by the inner stream, but at a greater distance. This last streamer is marked along its middle with brighter streaks. The main tail is bright and is full of curious structure. Its north side is strongly undulating. In the middle of the tail, about 2° from the head, is a long narrow curved bright strip, which is probably a streamer seen in projection, and which has the same kind of deflection shown by the other two.

I would call especial attention to this curvature in the streamers, because it is evident that no force existing either in the Sun or the head of the comet could produce them. It would seem to be due either to a repulsive force located in the main tail at these points, or to some disturbing medium located in that region of the sky.

Yerkes Observatory:
1908 *November* 28.

Observations of Comet c 1908, *from Photographs taken with the 30-inch Reflector of the Thompson Equatorial at the Royal Observatory, Greenwich.*

(*Communicated by the Astronomer Royal.*)

The following positions of Comet *c* 1908 were obtained from photographs taken with the 30-inch reflector. As a rule, two exposures of 2 minutes each were made on each plate.

The plates were measured with the astrographic micrometer. Six reference stars were measured with the comet, their positions being derived from the Greenwich Nine-Year Catalogue 1900, or the Catalogues of the Astronomische Gesellschaft.

The positions given are not corrected for Parallax.

Log Parallax Correction = log Parallax Factor − log Δ.

Date and G.M.T.				Apparent R.A.			Apparent Dec.			Log Parallax Factor.	
										R.A.	Dec.
1908.	d	h	m	s	h	m	s	°	′	″		
Sept.	4	14	30	12	3	15	26·88	+67	55	37·7	−9·653	−0·288
	4	14	56	51	3	15	22·42	67	56	16·3	−9·539	−0·339
	6	12	0	6	3	7	32·40	69	1	29·7	−9·953	−8·330
	7	12	43	48	3	2	32·87	69	37	58·4	−9·893	−0·025
	8	13	40	47	2	56	55·83	70	14	59·6	−9·751	−0·322
	14	9	25	41	2	10	53·78	73	40	5·2	−0·105	+9·913
	17	10	51	51	1	32	30·48	75	13	2·5	−9·996	−0·321
	18	11	36	57	1	16	42·72	75	38	46·1	−9·832	−0·480
	21	8	4	11	0	25	21·81	76	25	39·8	−0·151	−9·813
	25	8	35	20	23	0	45·24	76	3	34·3	−9·897	−0·457
	29	11	22	42	21	40	53·43	73	29	18·6	+9·855	−0·385
Oct.	1	11	43	8	21	10	5·54	71	29	8·9	+9·937	−0·104
	2	9	47	29	20	57	48·33	70	26	4·1	+9·642	−0·388
	3	14	5	9	20	43	39·03	68	58	29·4	+0·008	+0·432
	5	16	51	47	20	22	15·76	66	5	11·3	+9·747	+0·828
	6	9	16	2	20	16	19·11	65	5	28·2	+9·639	−0·154
	14	8	9	37	19	30	53·02	51	59	3·9	+9·506	+9·763
	15	7	37	27	19	27	23·28	50	15	53·8	+9·406	+9·777
	21	10	18	14	19	11	28·59	39	31	34·7	+0·664	+0·651
	23	7	48	4	19	8	4·50	36	17	34·4	+9·481	+0·492
	24	7	20	58	19	6	31·19	34	39	5·9	+9·423	+0·495
	26	8	42	14	19	3	40·23	31	17	46·3	+9·567	+0·644
	27	8	40	17	19	2	27·66	29	42	41·4	+9·564	+0·663
	29	6	8	24	19	0	26·44	26	47	36·7	+9·233	+0·587
	30	6	2	25	18	59	29·83	25	18	28·4	+9·223	+0·608
Nov.	3	6	13	39	18	56	26·23	19	40	35·6	+9·313	+0·688
	8	7	8	48	18	53	51·41	13	22	54·1	+9·468	+0·769
	9	6	44	5	18	53	28·66	12	15	4·0	+9·432	+0·770
	10	6	45	43	18	53	7·52	11	7	53·7	+9·441	+0·779
	17	6	13	1	18	51	26·49	4	7	7·1	+9·428	+0·817
	19	6	29	12	18	51	8·81	+ 2	19	44·0	+9·466	+0·827
	25	6	9	25	18	50	35·69	− 2	29	45·9	+9·472	+0·844

Royal Observatory, Greenwich:
1908 *December* 11.

S

N

New "Cave" Nebula in Cepheus.—Max Wolf.

A New "Cave-Nebula" in Cepheus. By Max Wolf, Ph.D. (Plate 6.)

I have the pleasure to announce the discovery of a new interesting nebula in the midst of the constellation of Cepheus.

The nebula was found by Dr. Kopff on a plate taken by him with my Bruce-telescope on the night of October 21st, 1908, and I photographed the object as soon as possible with our reflector. It has a very remarkable shape; and, as it forms an important addition to this class of nebulae, I forward the accompanying description.

The nebula involves the star B.D. +69°, 1231:

$$\alpha = 22^h\ 10^m\ 1^s, \quad \delta = +69^\circ\ 31'{\cdot}7 \quad (1855{\cdot}0),$$

which is given in the *B.D.* as 8·8 magnitude. This star was observed in three zones of the Christiania A.G. Catalogue where it has the number 3552 and is given as a star of magnitude 8·9, without any remark. The bright star visible south-west of the nebula on the photograph is B.D. +69°, 1228; it is given as 5·5 mag. in the same Catalogue. This nebula is a good example of the singular phenomenon of cave-formation amongst Milky Way stars. In some respects it shows the general characteristics of other cave-nebulæ, but it also offers several new features. It forms the end of a long starless lacuna, directed from south to north, resembling that of the T Cephei nebula. It has a bright condensation seen visually as a star of the ninth magnitude; it shows waves similar to those in the π_2 Cygni and T Cephei nebulæ, but has dark spaces north of the brighter parts, which seem darker than the sky in the neighbourhood. Round these dark spots extremely faint nebulous material is spread over the lacuna. The lacuna itself is about 1° to 2° long from south to north, and about 7′ to 10′ broad, but can be traced, with some interruptions, much further north. At about $22^h\ 10^m, +70^\circ{\cdot}0$ a bridge of stars traverses the long lacuna from east to west.

All over the cave lies a network of still darker spots and channels. This raises the hope that we may understand the interesting process more thoroughly at some future time, when we can photograph the region in more detail with greater optical power.

The bright southern edge of the nebula overlaps the region still filled with faint stars. So we find here, as already noted in some other objects, that the stars do not begin to decrease in number at the exact border of the visible nebula, but somewhat within it. The same was the case with the Z Orionis, the ψ Eridani, and other nebulæ. All such nebulæ have one border of great intensity, and this border overlaps the stars.

Some fine, complicated, but very small dark channels are visible in the south-western edge, near the stellar nucleus.

The reproduction given here is from a plate taken by me with the 28-inch Waltz-reflector of this observatory, November 16th, 1908, with two and a half hours' exposure through thin clouds.

Astrophysical Observatory, Heidelberg:
1908 *December* 4.

An Improved Telescope Triple Object-Glass.

By J. William Gifford. (Plate 7.)

(*Communicated by Sir David Gill, K.C.B.*).

The idea of employing three separate kinds of glass for the purpose of eliminating the secondary spectrum from the object-glasses of telescopes is not new. With the more limited choice of optical glass then possible, the problem was attacked successfully some thirty years ago by Sir Howard Grubb, Hastings, and Shroeder, and in more recent years by Abbe, Dennis Taylor, and Harting. Of the earlier workers, that of Hastings has come more immediately to my notice. Hastings draws attention (*Amer. Journ. Science and Arts*, vol. xviii., No. 108, p. 433) to the inaccurate measurements of the optical constants at his disposal, and attributes some anomalies in his results to this cause alone. Before proceeding further, it is my purpose to refer to a series of measurements of the optical constants of glasses which I have made for the purpose of comparison with those sent out by the makers, and from some of which the triple referred to and other lenses have been made.

The system adopted for making the measurements has already been described (*Roy. Soc. Proc.*, Feb. 13, 1902). Briefly, it is independent of measurements of angle. By means of a temperature refraction coefficient the indices are all brought to a temperature of 15° C., and by a new system of illumination greater accuracy in reading the goniometer circle is ensured. In all, 14 different glasses have been measured for the following wave-lengths:—

A′. 7682·45 (K_α)	E. 5270·11 (Fe)
B′. 7065·59 (He)	F. 4861·49 (H_β)
C. 6563·04 (H_α)	Φ. 4678·35 (Cd_6)
D. 5893·17 (Na)	G′. 4340·66 (H_γ)
A. 5607·1 (Pb)	

Seven of these have also been measured for wave-length H′ 3961·68 (Al), and the seven others for wave-lengths 4553·0 (Pt) 4415·0 (Cd).

By this we get a range of refractive indices for line D from 1·501118 (light Borosilicate crown) to 1·746410 (extra dense flint); and if the indices are placed in columns in ascending series by interpolating between corresponding elements in contiguous columns, it is easy to check any considerable errors in a glass whose optical position lies between those of the glasses to which the columns correspond.

It may be taken that all errors lower than the 5th. decimal place have been eliminated, and that in this place there are few greater than unity. The various lenses which have been calculated from these measurements, and since constructed by Hilger, have

all come out as accurately in agreement with the calculations as mechanical limitations allow.

In the case of most of the glasses referred to, after a preliminary rough measurement with a single prism, a part or all of the melting was purchased. Then, choosing three blocks of glass from it at hazard, these were cut into prisms, and each measured for some central line in the spectrum. The mean index thus obtained was used as a control for the others, in much the same way as the temperature coefficient, the others being brought each to a proportionate index.

To show that my indices are reliable, I have dwelt at some length on these points. I now hope to show that a slight variation in the index has a great, and in the case of a triple a profound, influence on the correspondence between the calculations and the performance of the lens after construction.

I take first the case of a $3\frac{1}{4}$-in. aperture doublet constructed some two years since. The glasses used were a light Borosilicate crown with $V = 63{\cdot}43$ and a dense flint with $V = 36{\cdot}75$. Lines C, D, and F were used for obtaining the correction for achromatism, lines C and F being equalised as to focal length. The makers' measurements and my own for these lines follow :—

Makers.	Gifford.
C. 1·50764	1·506254
D. 1·51000	1·508642
F. 1·51563	1·514265
Flint.	
C. 1·60710	1·606899
D. 1·61180	1·611638
F. 1·62367	1·623449

The foci for achromatism for a focal length of unity were—

Makers.	Gifford.
Crown 0·42155	0·42797
Flint −0·72877	−0·71812

Taking now, for the purpose of calculating a triple, a Baryta light flint with $V = 50{\cdot}25$, a Borosilicate flint with $V = 45{\cdot}82$, and a Borosilicate crown with $V = 60{\cdot}70$, and the same wave-lengths, we have for the refractive indices—

Baryta Light Flint.

Makers.	Gifford.
C. 1·57117	1·571003
D. 1·57450	1·574329
F. 1·58261	1·582433

Makers.	Gifford.
Borosilicate Flint.	
C. 1·57164	1·572056
D. 1·57530	1·575742
F. 1·58418	1·584621
Borosilicate Crown.	
C. 1·49875	1·498675
D. 1·50117	1·501118
F. 1·50700	1·506930

Passing over the tedious sum (shown in the Appendix for lines B′, Λ and Φ), the resulting foci for unity were—

Makers.		Gifford.
Baryta lt. ft.	0·37887	0·32402
Boro. ft.	−0·17443	−0·20618
Crown	0·24429	0·36179

It will be noted that my indices are taken to 6 places.* Now if we take these indices to 5 places only, the foci are as follows:—

Baryta Lt. ft. 0·31857, Boro ft. −0·21204, Boro crown 0·38805; and if we take them to 4 places only, then two of the foci become minus quantities.

All this goes to show how very sensitive the corrections of a triple for colour are to slight variations of refractive index, and why earlier constructors relied almost implicitly on figuring.

For the construction of the triple referred to in this paper, a 3-in. of 37·5-in. focus, the three above-mentioned glasses were taken, but instead of the usual corrections (lines B, D, and G being equalised), those chosen for the purpose were, lines B′7066·59, Λ5607·1, and Φ 4678·35. B′ is the brick-red line of helium, and is very near B.

Λ is a prominent lead line and occupies the position of maximum visual effect in the spectrum lying between D and E.

Φ is a cadmium line, and is near wave-length 4604 (Li), which has been mentioned by Sir W. Abney as the average point of maximum photographic effect for rays which have passed through any thickness of glass. As the Borosilicate flint is tinged with yellow, I have placed this point slightly lower.

Besides the evident advantage of using both Λ and this cadmium line, it was found that the use of Φ resulted in longer radii than if G or G′ were used, so that we are free from all the inherent difficulties of steep curves. Hastings lays down the point that no radius should be shorter than 1/15 of the focal length. This means that the curvature sums must not exceed 30 for a focus of 1.

* All the indices were observed to 7 places.

The following examples show the effect of using Φ instead of G′ (Hγ), and that the curvature sums are well within Hastings' requirements. Calling these curvature sums, with Hastings, A, B, and C, we have, with the lines for correction as shown,

B′, A and G′

A = 11·31376. B = − 12·45968. C = 3·34342.

B′, A and Φ

A = 8·94155. B = − 11·01247. C = 4·39960.

From these curvature sums the reciprocals of the foci were found. These reciprocals are really the angles made with the axis by each of the elements of the triple when the incident ray parallel to the axis is at such a height from the vertex that the emergent ray from the last surface makes one second of arc only with the axis, as described by Steinheil and Voit (*Handbuch der angewandten Optik*, p. 69), thus—

$$\frac{1}{f} = A(\mu - 1). \qquad \frac{1}{f'} = B(\mu' - 1). \qquad \frac{1}{f''} = C(\mu'' - 1).$$

A trigonometrical trace was then made for such a height, inserting these angles, and finding the corresponding radii at each surface. The difference at such a height between sine and arc is negligible (*loc. cit.*). From the final traces a diagram has been constructed for a lens of 19 metres focal length and a curve drawn. A similar curve for a doublet of the same focal length has also been drawn for comparison. It will be noted that the triple is almost independent of the wave-length.

With the radii and thickness found in this way the lens was then constructed, rigid adherence having been observed to the data given, without any attempt at improvement by figuring.

It was found, after squaring on, that although the definition was fair, a small amount of negative spherical aberration was present.

By separating the last two surfaces a few thousandths of an inch this disappeared, and a line test, such that at 608·7 yds. the angle subtended by the interval between any two of the lines was 1″·52 of arc, was resolved at 600 yds.

This is equivalent to a resolving limit of $\frac{4''\cdot55}{a}$.

It will be seen by this that the definition of this triple O.G of 3-in. aperture is excellent. The calculation for the final correction of colour having been made for rays incident at the extreme edge, any residual error is extremely difficult to detect with even high powers.

Mr. Nelson, on several occasions President of the R. Micro. Soc., to whom I should like to say I am indebted for all my

earlier teaching in optics and the trigonometrical trace of a ray, undertook the line test.

The actual radii and the thickness of the triple constructed are given in the Appendix, and also a worked-out example of the foci (thin lens) by a short formula which has been condensed from Coddington by Mrs. Gifford.

Before closing this paper, I should like to thank Sir David Gill (to whom my bringing this forward is entirely due) for his great kindness; Messrs. Chance Bros., Messrs. Schott of Jena, and Messrs. Mantois of Paris, for the facilities they have afforded me for measuring their glasses; and last, but not least, Mr. F. Twyman, of Messrs. Hilger Ltd., for the rare skill he has shown in accurately following the optical data given.

APPENDIX.

Reciprocals of Foci for Apochromatic Triple.

μ is baryta light flint. μ' is borosilicate flint. μ'' is borosilicate crown.

Differences of Proportional Spectra.

A′—D.	D—F.	F—G′.
0·00168	−0·00030	0·00805

Refractive Indices.

B′.7066(He).	A.5607(Pb).	Φ.4678(Cd$_6$).
$\mu_1 = 1{\cdot}569074$	$\mu_2 = 1{\cdot}576104$	$\mu_3 = 1{\cdot}584492$
$\mu'_1 = 1{\cdot}569909$	$\mu'_2 = 1{\cdot}577675$	$\mu'_3 = 1{\cdot}586869$
$\mu''_1 = 1{\cdot}497250$	$\mu''_2 = 1{\cdot}502400$	$\mu''_3 = 1{\cdot}508365$

$$A = \{(\mu_3-\mu_2)(\mu'_2-\mu'_1)-(\mu'_3-\mu'_2)(\mu_2-\mu_1)\}(\mu''_2-1) = 0{\cdot}00000025491 17.$$

$$B = \{(\mu''_3-\mu''_2)(\mu_2-\mu_1)-(\mu_3-\mu_2)(\mu''_2-\mu''_1)\}(\mu'_2-1) = -0{\cdot}0000007303256.$$

$$C = \{(\mu'_3-\mu'_2)(\mu''_2-\mu''_1)-(\mu''_3-\mu''_2)(\mu'_2-\mu'_1)\}(\mu_2-1) = 0{\cdot}0000005904547.$$

$$F = 1.$$

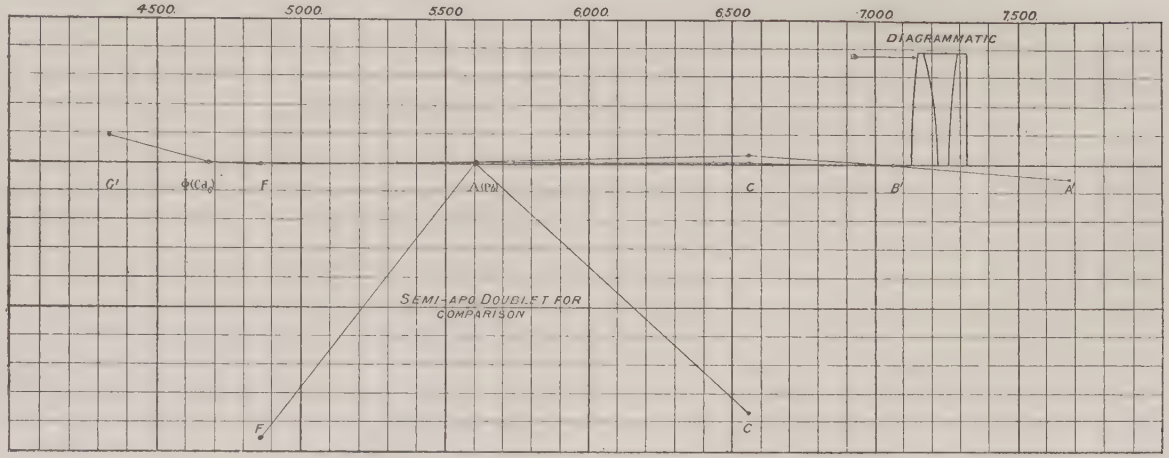
$$\frac{1}{f} = \frac{C}{A+B+C} = 5{\cdot}132568. \qquad \frac{1}{f'} = \frac{B}{A+B+C} = -6{\cdot}348405.$$

$$\frac{1}{f''} = \frac{A}{A+B+C} = 2{\cdot}215837.$$

$$\frac{1}{f} + \frac{1}{f'} + \frac{1}{f''} = 1. \qquad f' = -0{\cdot}157519.$$

$$\text{Corresponding curvature sum} = \frac{\frac{1}{f'}}{\mu'_2 - 1} = -10{\cdot}989579.$$

For the purposes of the curve, the data given have been multiplied by twenty. For comparison, the doublet already referred to has been treated similarly. All the focal lengths are from trigonometrical traces for edge rays, not for axial rays.



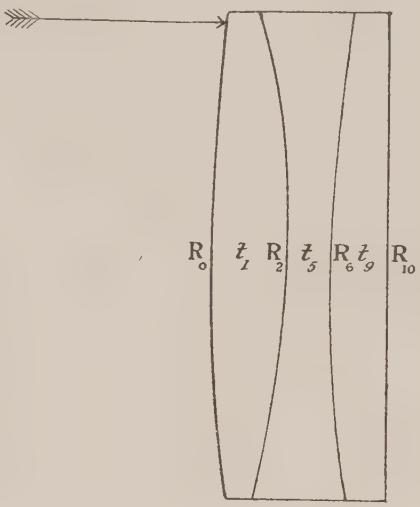
TRIGONOMETRICAL TRACE.

Optical Data by Trigonometrical Trace.

Aperture, 3-inch. Focus, 37·5-inch. Ratio, 12·5.

Radii.	Thicknesses.
$R_0 = 18{\cdot}28125$	$t_1 = 0{\cdot}450$
$R_2 = -5{\cdot}54507$	$t_3 = 0{\cdot}001$
$R_4 = R_2$	$t_5 = 0{\cdot}150$
$R_6 = 9{\cdot}13636$	$t_7 = 0{\cdot}003$
$R_8 = R_6$	$t_9 = 0{\cdot}350$
$R_{10} = -138{\cdot}755$	

Diagrammatic.



Note.—In order to get rid of the last trace of spherical aberration it became necessary to make R_4 0·001 shorter, and to alter R_{6-8} and R_{10} in proportion.

The following are the data calculated direct from the curvature sums without thickness, and starting from the same (arbitrary) radius R_0.

$R_0 = 18{\cdot}28125$
$R_2 = -5{\cdot}44237$
$R_4 = R_2$
$R_6 = 9{\cdot}09712$
$R_8 = R_6$
$R_{10} = -135{\cdot}180$

The influence of thickness, therefore, is to lengthen the radii.

It was thought that a few trigonometrical traces through another triple for ray A incident parallel to the axis at different heights from the vertex, and for other rays, would afford useful information with regard to the distance from the last surface along the axis at which each pencil intersects it, as follows :—

Ray.	Height from Vertex.	* Back Focus.	Focus.
A	0·00	28·98490	30·00000
,,	0·45	28·98496	30·00109
,,	0·90	28·98398	30·00425
,,	1·20	28·98330	30·00751
,,	1·35	28·98232	30·00906
,,	1·50	28·98142	30·01084
B′	0·00	28·99045	30·00140
Φ	0·00	28·97802	29·99812

* The distance of the point at which the ray intersects the axis from the last surface, along the axis.

These focal lengths are taken from another triple similar to the first, but having a ratio of aperture to focus of 10. They show that with spherical surfaces even a triple cannot be perfectly aplanatic.

Some focal lengths similarly traced of the 12·5 aperture ratio triple follow :—

Ray.	Height from Vertex.	Back Focus.	Focus.
A′	1·50	36·83056	37·49001
B′	,,	36·83988	37·50140
C	,,	36·84683	37·50739
Λ	0·00	36·84544	37·50000
,,	1·50	36·84157	37·50349
F	,,	36·83996	37·50354
Φ	,,	36·84110	37·50481
G′	,,	36·85882	37·52463

Fig. 1.

Fig. 2.

Note.—In order to construct a triple of the form shown in fig. 1 it is necessary to use two flints and a crown. The Steinheil type, fig. 2, is constructed by employing two crowns and a flint. One of these should be a heavy barium crown. The curves are steeper.

By using rays A′, Λ, and Φ as the basis of the colour correction, instead of B′, Λ, and Φ, the radii may be still further lengthened.

Data of Trigonometrical Trace.

For axial ray $H_0 = F$, for edge ray $H =$ height from vertex. Then for parallel incident ray, R_0 is arbitrary.

$$R_2 = \frac{(H_0 - t\delta_0)\mu - 1}{-\delta_2 + \delta_0(\mu - 1)}, \quad \delta_2 = \beta_{0,2} - \delta_0, \quad \beta_{0,2} = \frac{1}{f}^{*}. \quad R_4 = R_2$$

(or arbitrary).

$$R_6 = \frac{(A'_4 - t_5)\beta_{0,4}}{\theta_6 + \beta_{0,6}}, \quad \theta_6 = \frac{\delta_6}{1 - \mu'}, \quad \delta_6 = \beta_{4,6} - \delta_4, \quad \beta_{4,6} = \frac{1}{f}^{*}. \quad R_8 = R_6$$

(or arbitrary).

$$R_{10} = \frac{(A'_8 - t_8)\beta_{0,8}}{\theta_{10} + \beta_{0,8}}, \quad \theta_{10} = \frac{\delta_{10}}{1 - \mu''}, \quad \delta_{10} = \beta_{8,10} - \delta_8 \cdot \beta_{8,10} = \frac{1}{f'''}^{*}.$$

$t_1 = \alpha + \beta$, $\alpha = R_0 - \sqrt{R_0^2 - d^2}$, $\beta = R_2 - \sqrt{R_2^2 - d^2}$. $d =$ semi-aperture.

t_5 and t_9 are found similarly. t_3 and t_7, air thickness, are arbitrary.

* $\beta_{0,2}$ $\beta_{4,6}$ and $\beta_{8,10}$ are only approximately equal to $\frac{1}{f}$, $\frac{1}{f'}$ and $\frac{1}{f''}$; an adjustment must be made for thickness. This must be ascertained by the trigonometrical trace. The values in the example given were less by something near unity in the first decimal place.

Axial Ray.

$$\theta_0 = \frac{H_0}{R_0}$$

$$\phi_0 = \frac{\theta_0}{\mu}$$

$$\delta_0 = \theta_0 - \phi_0$$

$$A'_0 - R_0 = \frac{R_0\phi_0}{\delta_0}$$

$$\theta_2 = \frac{(A'_3 - R_2 - t_1)\delta_0}{R_2}$$

$$\phi_2 = \theta_2\mu$$

$$\delta_2 = \theta_2 - \phi_2$$

$$\beta_{0,2} = \delta_0 + \delta_2$$

$$A'_2 - R_2 = \frac{R_2\phi_2}{\beta_{0,2}}$$

$$\theta_4 = \frac{(A'_2 - R_4 - t_3)\beta_{0,2}}{R_4}$$

$$\phi_4 = \frac{\theta_4}{\mu}$$

$$\delta_4 = \theta_4 - \phi_4$$

$$\beta_{0,4} = \delta_4 + \beta_{0,2}$$

$$A'_4 - R_4 = \frac{R_4\phi_4}{\beta_{0,4}}$$

$$\theta_6 = \frac{(A'_4 - t_5 - R_6)\beta_{0,4}}{R_6}$$

$$\phi_6 = \theta_6\mu'$$

$$\delta_6 = \theta_6 - \phi_6$$

$$\beta_{0,6} = \delta_6 + \beta_{0,4}$$

$$A'_6 - R_6 = \frac{R_6\phi_6}{\beta_{0,6}}$$

$$\theta_8 = \frac{(A'_6 - R_8 - t_9)\beta_{0,6}}{R_8}$$

$$\phi_8 = \frac{\theta_8}{\mu''}$$

$$\delta_8 = \theta_8 - \phi_8$$

$$\beta_{0,8} = \delta_8 + \beta_{0,6}$$

$$A'_8 - R_8 = \frac{R_8\phi_8}{\beta_{0,8}}$$

$$\theta_{10} = \frac{(A'_8 - R_{10} - t_9)\beta_{0,8}}{R_{10}}$$

$$\phi_{10} = \theta_{10}\mu''$$

$$\delta_{10} = \theta_{10} - \phi_{10}$$

$$\beta_{0,10} = \delta_{10} + \beta_{0,8}$$

$$A'_{10} - R_{10} = \frac{R_{10}\phi_{10}}{\beta_{0,10}}$$

$$A'_{10} = (A'_{10} - R_{10}) + R_{10}$$

$$F_{0,10} = \frac{H_0}{\beta_{0,10}}$$

Edge Ray.

$$\sin\theta_0 = \frac{H}{R_0}$$

$$\sin\phi_0 = \frac{\sin\theta_0}{\mu}$$

$$\delta_0 = \theta_0 - \phi_0$$

$$A'_0 - R_0 = \frac{R_0\sin\phi_0}{\sin\delta_0}$$

$$\sin\theta_2 = \frac{(A'_0 - R_2 - t_1)\sin\delta_0}{R_2}$$

$$\sin\phi_2 = \mu\sin\theta_2$$

$$\delta_2 = \theta_2 - \phi_2$$

$$\beta_{0,2} = \delta_0 + \delta_2$$

$$A'_2 - R_2 = \frac{R_2\sin\phi_2}{\sin\beta_{0,2}}$$

etc., etc., sines of angles being substituted for angles where shown.

Notes.

A'_{10} is the distance of the last surface, along the axis, from the point at which the emergent ray intersects the axis.

$F_{0,10}$ is the distance of the optical centre of the triple, along the axis, from the point above named.

In the working for axial ray, $\beta_{0,10}$ = one second of arc in every case.

On the Determination of the apparent Diameter of a fixed Star.

By Major P. A. MacMahon, D.Sc., F.R.S.

Some sort of an idea of the size of a fixed star may be formed from a knowledge of its parallax and of its magnitude, but astronomers have not felt justified in placing much reliance upon any estimate obtained in this manner. A comparison between the light of the Sun and that of a star is difficult because of the immense difference between them. Statements ascribing to α Centauri and Sirius intrinsic splendours equal to 2·3 and 63 times that of the Sun respectively can only be regarded as indications that the truth is probably somewhere in the neighbourhood of these numbers.

Whether these differences of splendour are due to differences of diameter or of the amount of light emitted by equal areas requires further observation, and a direct method of determining a star's apparent diameter is much to be desired. For this purpose those stars which periodically experience occultation by the Moon are the most favourably situated for investigation. It will be remembered that certain stars were first shown to be double by peculiarities observed during occultation, and the idea comes into the mind that it may be possible to subject the phenomenon of lunar occultation to a refined and expanded photographic examination by employment of the long, rapidly moving films which are used in the modern bioscope.

The time taken by the Moon to occult a fixed star has been described as "instantaneous," or "less than the twinkling of an eye," and in other terms also indicating the opinion that the time in question is at once inappreciable and immeasurable. Very minute intervals of time are, however, measurable. Sir Andrew Noble's chronograph measures $\frac{1}{1000000}$ of a second. Professor Worthington has recently photographed the splash formed by a drop of water in a time estimated to be $\frac{3}{1000000}$ second, which does not differ much from a light-kilometre.

The question then arises as to the time probably taken by the Moon to occult an average fairly bright star.

I had made a few calculations based upon parallaxes which astronomers regard as being reliable within a small probable error, and had found that the time might be $\frac{1}{1000}^{s}$, when I took the opportunity of consulting Professor Dyson. He assumed the star to have a parallax $0''{\cdot}10$, and to be of the same actual diameter as the Sun; this leads to an apparent diameter $\frac{1}{1000}''$, and the time taken by the Moon to effect a direct occultation would then be about $\frac{1}{500}^{s}$.

In a letter to me, dated 1908 November 1, Professor Dyson proceeds: "Thus exposures of $\frac{1}{1000}^{s}$ or rather less would be required. With the Astrographic (13 in. O.G.) telescope a star of $9^{m}{\cdot}0$ registers so as to give a measurable image in 10^{s}. A star of

$4^{m}\cdot0$ would be shown in $\frac{1}{100}$ of this, or $\frac{1}{10}^{s}$. With a telescope like the 30-inch reflector at Greenwich the time of exposure could be shortened about 6 times, and this is as far as can be got—so that it is possible that a star of $1^{m}\cdot0$ might be photographed in $\frac{1}{15} \times \frac{1}{6} \times \frac{1}{10}$, or $\frac{1}{780}^{s}$. These figures are, of course, hypothetical; but with the largest telescopes a few of the brightest stars might be photographed in $\frac{1}{300}^{s}$, I should think; and if the diameter were three or four times that of the Sun, a result might be got . . . but, as far as I can see, the limits within which the method is feasible will depend upon the size of the telescope and the sensitiveness of the plate, and not upon the mechanical difficulty of giving the exposures. It will only be necessary to move the plate so that a trail will be shown."

If we assume that the apparent diameter of the star is something like $\frac{1}{1000}''$, the time taken by the Moon to occult it would be about $\frac{1}{500}^{s}$ if the star were in the line of motion of the Moon's centre. This direct occultation is the most unfavourable for the present purpose. In general, the occultation would be oblique and the time taken to occult longer.

If the radius of the Moon which passes through the star make an angle β with the direction of the Moon's motion, the time for the occultation to be effected would be $\sec\beta$ times that of direct occultation, as may be readily verified. Thus if $\beta = 60^{\circ}$, the time would be doubled; if $\beta = 82^{\circ}\ 50'$ the time would be multiplied by 8, and so on.

Assuming the Moon to have an unbroken outline, occultation is not possible if β exceeds a certain magnitude. If the Moon all but misses the star, the time taken to effect the occultation will be a maximum, and equal to

$$\sqrt{\frac{D}{d}} \cdot \tau$$

where D, d are the apparent diameters of the Moon and star respectively, and τ is the time in respect of direct occultation. This is, in fact, the maximum value of $\tau \sec\beta$; d being small compared with D. If it were possible to observe such an occultation, the time to be dealt with might be as great as one or two seconds. Failing this, favourable conditions might well give a time longer than $\frac{1}{100}^{s}$.

Can this time be measured either by a moving film giving a trail, or by the separate exposures of the kinematograph?

On account of the rough edge of the Moon it is probable that many different occultations of the same star would have to be observed before information in regard to its apparent diameter could be deduced.

On some points with regard to the Light-Fluctuation of Variabl Stars. By Karl Pearson, F.R.S., assisted by Julia Bell, M.A. University College, London. (Plates 8–10.)

(1) *Introductory.*—In the course of last session's statistica lectures I worked out, as an illustrative example of method, th correlation between light fluctuation and period in the case o short-period variable stars; and, having regard to Professor Turner' recently published paper,* it may not be without interest to indi cate one or two further points in the same field of inquiry.

Some preliminary remarks are necessary. Let L be the light o a star of magnitude m. Then if the unit of light be the light of a star of the 10th magnitude,†

$$L = 10000/a^m$$

where $\log a = \cdot 4$, using Pogson's value.

If L_2 be the light of a star at the maximum and L_1 at th minimum, m_2 and m_1 the corresponding magnitudes:

$$\log L_2 = 4 - \cdot 4m_2,\ \log L_1 = 4 - \cdot 4m_1 \quad . \quad . \quad (i$$

and

$$\log (L_2/L_1) = \cdot 4(m_1 - m_2) \quad . \quad . \quad . \quad (ii$$

whence we easily calculate $(L_2 - L_1)/L_2 = F$, which I term th light-fluctuation. This measures *relative* light change, and is fre of distance and other factors, which is not the case with $(m_1 - m_2)/m_2$ I desire first to indicate certain general results. Let σ_{L_2}, σ_{L_1}, $\sigma_{L_2-L_1}$ be the standard deviations of maximum light, mimimum light, and range of light variation. Then if r_{L_1, L_2} be the correlation between maximum and minimum amounts of light, and $r_{L_2-L_1, L_2}$ the correla tion of light range and maximum light, we have

$$\sigma^2_{L_2-L_1} = \sigma^2_{L_2} + \sigma^2_{L_1} - 2\sigma_{L_2}\sigma_{L_1}r_{L_1L_2} \quad . \quad . \quad . \quad (iii$$

$$\sigma^2_{L_1} = \sigma^2_{L_2-L_1} + \sigma^2_{L_2} - 2\sigma_{L_2-L_1}\sigma_{L_2}r_{L_2-L_1, L_2} \quad . \quad . \quad (iv$$

whence

$$r_{L_1, L_2} = \frac{\sigma^2_{L_2} + \sigma^2_{L_1} - \sigma^2_{L_2-L_1}}{2\sigma_{L_2}\sigma_{L_1}} \quad . \quad . \quad . \quad . \quad . \quad (v$$

$$r_{L_2-L_1, L_2} = \frac{\sigma_{L_2} - \sigma_{L_1}r_{L_2L_1}}{\sigma_{L_2-L_1}} \quad . \quad . \quad . \quad . \quad . \quad (vi$$

Thus (v) enables us to find the correlation between maximum and minimum luminosities, and (vi) between maximum luminosity and range of luminosity.‡ The above results are, of course,

* *M.N.*, June 1908, vol. lxviii. p. 544.

† It would be convenient to have a name for this unit of stellar light, and perhaps *stellace* = star-unit (*stella* and *as*) would serve.

‡ Range of luminosity $L_2 - L_1$ must be carefully distinguished from the light-fluctuation $(L_2 - L_1)/L_2$ defined above.

absolutely true without any reference to the nature of the frequency distribution. They are algebraical identities. Similarly we have

$$r_{m_1,\,m_2} = \frac{\sigma_{m_1}^2 + \sigma_{m_2}^2 - \sigma_{m_1-m_2}^2}{2\sigma_{m_1}\sigma_{m_2}} \quad . \quad . \quad . \quad . \quad \text{(vii)}$$

$$r_{m_1-m_2,\,m_2} = \frac{\sigma_{m_1} r_{m_1,\,m_2} - \sigma_{m_2}}{\sigma_{m_1-m_2}} \quad . \quad . \quad . \quad . \quad \text{(viii)}$$

Now the point I desire to emphasise here is that $(\sigma_{m_1} r_{m_1,\,m_2} - \sigma_{m_2})$ may be very small, and so the correlation $r_{m_1-m_2,\,m_2}$ very small, and yet $r_{L_2-L_1,\,L_2}$ may be very large indeed. In other words, to show that there is very little relationship between magnitude at the maximum and range of magnitude is not to demonstrate that close relationship between maximum luminosity and range of luminosity is wanting. For example, in one series of short-period variables I find $r_{m_1-m_2,\,m_2} = \cdot 20$, but $r_{L_2-L_1,\,L_2} = \cdot 996$. Further, to demonstrate that $r_{L_1,\,L_2}$ is small is not to demonstrate that $r_{L_2-L_1,\,L_2}$ is necessarily small, because (vi) shows that $r_{L_2-L_1,\,L_2}$ would have a finite value if $r_{L_1,\,L_2}$ were zero. Thus in a series of short-period variables we have

$$r_{L_2,\,L_1} = \cdot 25, \text{ but } r_{L_2-L_1,\,L_2} = \cdot 93.$$

It seems to me absolutely needful to use the true physical variable, amount of luminosity, and not the merely physiological variable magnitude, in any inquiry of this kind. It has already been indicated that a marked difference arises when parallax is correlated with magnitude instead of with luminosity,* and it may not be without interest to ascertain by a rough process how the difference arises in a case like the present. We shall obtain approximate formulæ, often sufficiently accurate, by assuming the variations are small as compared to the means. Thus we may use differentials

$$\log \mathrm{L} = 4 - \cdot 4m$$

$$\frac{\delta \mathrm{L}}{\bar{\mathrm{L}}} = -\cdot 4\delta m$$

where $\bar{\mathrm{L}}$ is the mean amount of light and may here be put for L.

Accordingly

$$\delta \mathrm{L}_2 - \delta \mathrm{L}_1 = (\bar{\mathrm{L}}_1 \cdot 4\delta m_1 - \bar{\mathrm{L}}_2 \cdot 4\delta m_2)$$

$$\delta \mathrm{L}_2 = -\cdot 4\bar{\mathrm{L}}_2 \delta m_2.$$

Thus

$$\sigma_{L_2-L_1}\sigma_{L_2} r_{L_2-L_1,\,L_2} = \cdot 16(\bar{\mathrm{L}}_2^2 \sigma_{m_2}^2 - \bar{\mathrm{L}}_1 \bar{\mathrm{L}}_2 \sigma_{m_2}\sigma_{m_1} r_{m_1,\,m_2}),$$

and $\sigma_{\mathrm{L}_2}^2 = \cdot 16\bar{\mathrm{L}}_2^2\sigma_{m_2}^2$, $\sigma_{L_2-L_1}^2 = \cdot 16(\bar{\mathrm{L}}_1^2\sigma_{m_1}^2 + \bar{\mathrm{L}}_2^2\sigma_{m_2}^2 - 2\bar{\mathrm{L}}_1\bar{\mathrm{L}}_2\sigma_{m_1}\sigma_{m_2} r_{m_1,\,m_2})$.

* *M.N.*, vol. lxvi. pp. 447, 450. I venture to suggest that possibly additional knowledge might be gained as to the nature of variables by considering the light-curves as well as the magnitude curves.

Thus

$$r_{L_2-L_1,\,L_2} = \frac{\bar{L}_2\sigma_{m_2} - \bar{L}_1\sigma_{m_1}r_{m_1,\,m_2}}{\sqrt{\bar{L}_1{}^2\sigma^2_{m_1} + \bar{L}_2{}^2\sigma^2_{m_2} - 2\bar{L}_1\bar{L}_2\sigma_{m_1}\sigma_{m_2}r_{m_1,\,m_2}}} \quad . \quad (ix)$$

By (viii.)

$$r_{m_1-m_2,\,m_2} = \frac{\sigma_{m_1}r_{m_1,\,m_2} - \sigma_{m_2}}{\sqrt{\sigma^2_{m_1} + \sigma^2_{m_2} - 2\sigma_{m_1}\sigma_{m_2}r_{m_1,\,m_2}}} \quad . \quad . \quad . \quad (x)$$

To reason on $r_{m_1-m_2,\,m_2}$ as if it were $r_{L_2-L_1,\,L_2}$, or proportional to it, is to omit all allowance for the distorting factors $\bar{L}_2$, $\bar{L}_1$. The short-period variable series which gives me $r_{m_1-m_2,\,m_2} = \cdot 20$ from (x), gives me $r_{L_2-L_1,\,L_2} = \cdot 955$, as found from the merely approximate formula (ix), against $\cdot 996$ as found from the absolutely accurate formula (vi). We must, I think, conclude from these results that it is needful to argue concerning amounts of light, and not magnitudes, when we are applying our conclusions to the consideration of the physical conditions directly controlling the light conditions.

Another point deserves consideration at this stage. I have indicated that *mathematically* a correlation exists between L_2 and $L_2 - L_1$, even if there be no correlation between L_1 and L_2 at all. This fact is spoken of in modern statistics as *spurious* correlation, a correlation arising purely from an arithmetical process.* But the fact that, if we took two *random* series of numbers and subtracted one from the other, this would give us a correlation between either series and the series of differences is not applicable, as a little consideration will show, to the present case. For the fundamental physical variables are not the maxima and minima amounts of light L_2 and L_1, but the amount of light and some measure of the obscuring cause O, and it will be seen at once that using $L_2 - L_1$ as a measure of O is a natural process, and not an arithmetical differencing of two separate physical conceptions. We are throughout concerned with the relation of the amount of obscuration to the full light of the star. If we were correlating—to take an illustration—the average sun-spot area with the total visible area in a population of suns, we should not, I think, attribute the correlation to a mere arithmetical process if we had reached our results by measuring on all occasions the total bright area, and subtracting variations in the extent of that area. In other words $L_2 - L_1$ is to be looked upon as a fundamental character of the variable, *i.e.* as a measure of the obscuring cause, and not as an artificial difference of two physical characters resulting in spurious correlation. A high correlation between O and L_2, if it exists, represents something physical, not the result of an arithmetical process.†

It has been further suggested that possibly in the case of certain classes of variables a star would only be entered if $L_2 - L$

* Pearson, "On Spurious Correlation," *R.S. Proc.*, vol. 60, pp. 489 *et seq.*

† Pearson, "Influence of Selection on Variability and Correlation," *Phil. Trans.*, vol. 200, A, pp. 1–66.

or O were large, relative to L_2. But it is difficult to see how such a process of selection would tend to produce a high correlation between O and L_2. Correlation is a relation between the *deviation* in O and the *deviation* in L_2 from their mean values; the process indicated is one which would raise the mean value of O, and possibly lessen its variability. The general theory of correlation shows that such selection *reduces* the true correlation, and does not spuriously increase it.*

I insert these remarks in order to meet criticisms which I think may readily occur to those not accustomed to the various aspects of correlation problems. I now turn to the subject in hand.

(2) We may suppose the light of a star as it reaches us to be a function of three quantities: (i) the chemical constitution, C; (ii) the size, S; and (iii) the distance, D, from the observer. Hence let us put

$$L = \gamma.C.S./D^2,$$

where γ is a scale constant.

Of these factors it seems unlikely that distance is correlated with chemical constitution or size, so that we may put $r_{C,D}$ and $r_{D,S} = 0$. There is some possibility that C and S are correlated, although it is conceivable that there should be various stars of the same chemical constitution of different sizes. In a first consideration we may consider $r_{C,S}$ small, and neglect it to a rough approximation. Various other correlations could in the present state of our knowledge be directly found. Thus $r_{L,C}$ is the correlation of luminosity and chemical constitution, and would be given by correlating magnitude and spectral class, provided we proceeded by the method of contingency, and not by that of direct correlation. It has been shown to be of the order ·6 to ·7.† $r_{L,S}$ is the correlation of luminosity and size, and we might hope in time to attempt its evaluation, as our knowledge of the parallaxes of binaries increases.‡ $r_{L,D}$ is the magnitude or luminosity and parallax correlation, and its value is probably about ·3. Thus eventually we may well be in a position to determine the relation of L to all its constituent factors. At present we are not in a condition to achieve this, but I think we may be able in a rough sort of way to express some general suggestions as to the relation between the obscuring cause and the size factor at full luminosity. Let S_2 be the size factor at maximum luminosity, $S_2 - S_1$ at minimum luminosity. Then S_1 marks the amount of obscuration; and in using these symbols we pledge ourselves to no theory of the source

* A man's income, *i*, being paid annually, his balance, *b*, at the end of the year gives his expenditure $e = i - b$, but the correlation between income and expenditure, *i* and *e*, is not due to the appearance of *i* in the mathematical forms *i* and $i - b$; the latter is only a convenient way of finding the value of *e*.

† *M.N.*, vol. lxviii. p. 423 *et seq.*

‡ The few masses of double stars with which I am acquainted suggest an extraordinarily high value for $r_{L,S}$, or again emphasise the point that distance plays a relatively small part in determining L.

of the obscuration; it may be due to partial eclipse by a dark companion; to partial eclipse of a bright companion by a bright primary; to the phases of a dark companion without eclipse; to something of the nature of sun-spots, or to a non-spherical body rotating round an axis of figure. In all such cases, and in other conceivable ones, we can give a physical interpretation to S_1.

We have at once

$$L_2 = \gamma C_2 S_2 / D^2 \quad . \quad . \quad . \quad . \quad \text{(xi)}$$

$$L_1 = \gamma C_2 (S_2 - S_1) D^2 \quad . \quad . \quad . \quad \text{(xii)}$$

taking D to remain sensibly the same. Hence

$$L_2 - L_1 = \gamma C_2 S_1 / D^2 \quad . \quad . \quad . \quad . \quad \text{(xiii)}$$

and

$$F = S_1 / S_2 \quad . \quad . \quad . \quad . \quad . \quad \text{(xiv)}$$

A study of F will therefore tell us something about the ratio of the area of obscuration to the size factor. In particular, in the case of the eclipse type of variable, we shall have some idea of the relation between the area of the eclipsed and eclipsing bodies.

Additional notation that will be needful is the use of a bar to denote the mean value of a variable, and the use of the letter V to mark a coefficient of variation, *i.e.* the ratio of the standard deviation of a quantity to its mean value. For example,

$$V_{S_2} = \sigma_{S_2} / S_2.$$

Thus, V_{S_2} is the variability of the size factor relatively to the mean value of the size factor. Assuming that to a first approximation we may represent deviations from the mean by differentials and further that $r_{C,D}$, $r_{S,D}$ and what is more questionable $r_{S,C} = 0$, we obtain easily the following formulæ:

$$V^2_{L_2} = V^2_{C_2} + V^2_{S_2} + 4V^2_D \quad . \quad . \quad . \quad \text{(xi}$$

$$V^2_{L_2 - L_1} = V^2_{C_2} + V^2_{S_1} + 4V^2_D \quad . \quad . \quad . \quad \text{(x}$$

Hence

$$V^2_{L_2} - V^2_{L_2 - L_1} = V^2_{S_2} - V^2_{S_1} \quad . \quad . \quad . \quad \text{(xv}$$

Now it is quite easy to determine V_{L_2} and $V_{L_2 - L_1}$ from the data. We are thus in a position to discover whether the relative variability in size of the fully lighted surface or of the obscuration is the larger. Further it follows at once that

$$V_{L_2 - L_1} V_{L_2} r_{L_2 - L_1, L_2} - V^2_{L_2 - L_1} = V_{S_1} V_{S_2} r_{S_1, S_2} - V^2_{S_1} \quad . \text{(xv}$$

$$V_{L_2 - L_1} V_{L_2} r_{L_2 - L_1, L_2} - V^2_{L_2} = V_{S_1} V_{S_2} r_{S_1, S_2} - V^2_{S_2} \quad . \quad . \text{(xv}$$

I believe that (xvii) and (xviii), which involve (xvi), are the only relations of an independent kind between the $L_2 - L_1$ and

constants on the one hand, and the areas of maximum luminosity (S_2) and of maximum obscuration (S_1) on the other hand, which do not involve the chemical constitution or distance factors C or D.

It will be at once obvious that if r_{S_1, S_2} be zero, or there be no relation between the sizes of the areas of maximum luminosity and of maximum obscuration, the left-hand sides of (xvii) and (xviii), which are numerically determinable, must come out negative for such cases. If, however, either (xvii) or (xviii) give a positive left-hand side, it must signify that r_{S_1, S_2} is sensibly positive, and we may be able from the two equations, or either and (xvi), to set a limit to the minimum value possible of r_{S_1, S_2}.

(3) *Material used.*—I have used three types of variable stars :—

(i) *Short-Period Cluster Variables.*—I owe to Professor Turner the reference to the homogeneous material provided by Bailey.* There are 91 stars in his list, all, with one exception, having a period under one day.

(ii) *Short-Period Algol Type Variables.*—I have used the series of 57 stars provided in the *Companion to the Observatory* for 1908, p. 26. On the basis of this series, I had found out that a considerable correlation existed between period and light fluctuation, before I consulted Professor Turner as to additional material. The disadvantage of this list of Algol type stars lies in the fact that *Algol*, λ *Tauri*, and V *Puppis* are so luminous compared with the remaining 54 stars that they influence out of all proportion some of the constants. For example, the mean light-range $L_2 - L_1$ and the mean light of the bright star L_2, including these three stars are 25·03 and 40·56 respectively, but excluding them, they fall to 5·31 and 8·22 respectively ! My reductions have therefore been made with and again without these three stars.

(iii) *Long-Period Variables.*—I have taken 87 stars out of Chandler's Third Catalogue.† The stars have been selected by rejecting those of indefinite minima, *i.e.* with a minimum stated to be "less than a certain magnitude." Further, the average has been taken of the maximum values. A glance at Table IV. will show that there are eight stars asterisked in that table with light fluctuations under ·8. These stars possibly belong to heterogeneous material. Their peculiarity is well evidenced on Diagram III., wherein it is shown that the 79 stars lie almost exa tly on a line, and how much this line is shifted when we include the eight. The chief constants have been worked out in duplicate, with and without the eight. Looking at the light (or as I think it would be better to call them magnitude) curves of these three groups, we recognise very marked distinctions. It is not the object of the present paper to inquire into these distinctions, but to ask whether there are any marked differences in the statistical constants of the three groups which would suffice to indicate marked differences between their physical characters apart from the light curves. We shall use

* "Variable Stars in the Cluster ω *Centauri*," by Solon I. Bailey, *Harvard Annals*, xxviii. pp. 133–5.

† *Astronomical Journal*, No. 379, vol. xvi. pp. 145–172.

only for our present purposes the language of L_2, L_1, S_2 and S_1; the exact interpretation of S_1 must depend on what theory is adopted of the source of the light variation. If we are dealing with the Algol type group, it is a measure of the area eclipsed and roughly of the size of the eclipsing body; in the case of the long-period variables, it may be of the nature of a maximum spot area. Suggestions of the nature of S_1 have yet to be made for the case of the short-period cluster variables.

(4) *Short-Period Cluster Variables.*—The constants for this group are as follows, the unit of light being a stellace:

Mean light at maximum	$\bar{L}_2 = \cdot 0572$
Variability of light at maximum	$\sigma_{L_2} = \cdot 0137$
Mean light at minimum	$\bar{L}_1 = \cdot 0258$
Variability of light at minimum	$\sigma_{L_1} = \cdot 0051$
Mean light range	$\bar{L}_2 - \bar{L}_1 = \cdot 0314$
Variability of range	$\sigma_{L_2 - L_1} = \cdot 0134$

The coefficients of variability are:

$$V_{L_2} = \cdot 2399,\ V_{L_1} = \cdot 1984,\ V_{L_2 - L_1} = \cdot 4237.$$

The coefficients of correlation are:*

$$r_{L_1, L_2} = \cdot 2512$$

$$r_{L_2 - L_1, L_2} = \cdot 9283$$

It is easy to show, on the hypotheses stated above (p. 132) that,

$$V_{L_2 - L_1} V_{L_2} r_{L_2 - L_1, L_2} = V_C^2 + 4V_D^2 + V_{S_1} V_{S_2} r_{S_1, S_2} \quad . \qquad \text{(xix)}$$

$$V_{L_2} V_{L_1} r_{L_1, L_2} = V_C^2 + 4V_D^2 + \frac{\sigma_{S_2}^2 - \sigma_{S_1}\sigma_{S_2} r}{\bar{S}_2(\bar{S}_2 - \bar{S}_1)^2} \quad . \qquad \text{(xx)}$$

If we suppose $r_{S_1, S_2} = 0$, then we have,

$$r_{L_2 - L_1, L_2} = (V_C^2 + 4V_D^2)/V_{L_2 - L_1} V_{L_2}$$

$$r_{L_1 L_2} = \{V_C^2 + 4V_D^2 + V_{S_2}^2 \bar{S}_2/(\bar{S}_2 - \bar{S}_1)\}/V_{L_1} V_{L_2}.$$

* These were found from Equations (v) and (vi). The remaining constants were determined without grouping, by direct square and produc deviations from mean values, as is better for these short series.

TABLE I.

Taken from Annals of Harvard College Observatory, 1902.
Paper by S. I. Bailey.

Index No.	Period in Days.	Light-Fluctuation.	Index No.	Period in Days.	Light-Fluctuation.	Index No.	Period in Days.	Light-Fluctuation.
3	·84	·52	44	·57	·54	85	·74	·54
4	·63	·66	45	·59	·69	86	·65	·68
5	·52	·70	46	·69	·61	87	·40	·37
7	·71	·63	47	·49	·39	88	·69	·48
8	·52	·74	49	·60	·60	89	·37	·43
9	·52	·70	50	·39	·40	91	·90	·56
10	·38	·41	51	·57	·67	96	·62	·54
12	·39	·37	52	·66	·50	97	·69	·60
13	·67	·65	54	·77	·46	99	·77	·60
14	·38	·48	55	·58	·56	100	·55	·52
15	·81	·56	59	·52	·53	101	·34	·34
18	·62	·68	60	1·35	·63	102	·69	·56
19	·30	·51	62	·62	·62	104	·87	·39
20	·62	·71	63	·83	·43	105	·34	·43
21	·62	·37	64	·34	·39	106	·57	·62
22	·40	·39	66	·41	·37	107	·51	·65
23	·51	·66	67	·56	·58	108	·59	·43
24	·46	·40	68	·53	·35	113	·57	·70
25	·59	·63	69	·65	·64	115	·63	·65
26	·78	·54	70	·39	·35	117	·42	·40
27	·62	·47	71	·36	·46	118	·61	·68
30	·40	·34	72	·38	·37	120	·55	·62
32	·62	·62	73	·58	·67	122	·63	·65
33	·60	·67	74	·50	·71	123	·32	·37
34	·73	·56	75	·42	·37	124	·33	·43
35	·39	·48	76	·34	·37	125	·59	·67
36	·38	·32	77	·43	·34	126	·34	·40
38	·78	·63	79	·61	·68	130	·49	·73
39	·39	·43	81	·39	·48	132	·66	·63
40	·63	·62	82	·34	·37			
41	·66	·65	83	·36	·43			

Now $V_{L_2-L_1}$ is twice as great as V_{L_1}, and $\bar{S}_2 - \bar{S}_1$ is by the nature of the case positive. It therefore follows that if r_{S_1, S_2} were zero we should have r_{L_1, L_2} more than twice as large as $r_{L_2-L_1, L_2}$; as a matter of fact $r_{L_2-L_1, L_2}$ is more than three and a half times as large as r_{L_1, L_2}. This is only possible if r_{S_1, S_2}, so far from being zero, has

a very substantial value. I think we have here evidence that the size of the area of obscuration must be closely correlated with the size of the star itself.

We may approach the question also from (xvi). We have

$$V_{S_2}^2 - V_{S_1}^2 = (\cdot 2399)^2 - (\cdot 4237)^2 = -\cdot 1220 \quad . \quad \text{(xxi)}$$

It is clear, therefore, that V_{S_2} is less than V_{S_1}, or that the relative variability in the area S_1 of obscuration is greater than the relative variability of luminosity of the star itself.

We shall now consider equation (xviii). We have

$$\cdot 2399(\cdot 4237 \times \cdot 9283 - \cdot 2399) = V_{S_1} V_{S_2} r_{S_1, S_2} - V_{S_2}^2$$

or

$$\cdot 0368 = V_{S_2}(V_{S_1} r_{S_1, S_2} - V_{S_2}) \quad . \quad . \quad \text{(xxii)}$$

It is thus seen, again, that it is impossible for r_{S_1, S_2} to be zero. Eliminate V_{S_1} between (xxi) and (xxii), writing for generality:

$$\left.\begin{aligned} n_1 &= V_{S_1}^2 - V_{S_2}^2 \\ n_2 &= V_{S_2}(V_{S_1} r_{S_1, S_2} - V_{S_2}) \end{aligned}\right\} \quad . \quad . \quad \text{(xxiii)}$$

We have

$$V_{S_2}^4(1 - r_{S_1, S_2}^2) + 2V_{S_2}^2(n_2 - \tfrac{1}{2} n_1 r_{S_1, S_2}^2) + n_2^2 = 0 \quad . \quad \text{(xxiv)}$$

Thus for real roots, which is an essential condition, we must have

$$(n_2 - \tfrac{1}{2} n_1 r_{S_1, S_2}^2)^2 > (1 - r_{S_1, S_2}^2) n_2^2$$

$$r_{S_1, S_2}^2 \{\tfrac{1}{4} n_1^2 r_{S_1, S_2}^2 - n_2(n_1 - n_2)\} > 0$$

which leads to

$$r_{S_1, S_2} > \frac{2}{n_1} \sqrt{n_2(n_1 - n_2)} \quad . \quad . \quad . \quad \text{(xxv)}$$

In the present case this gives

$$r_{S_1, S_2} > \cdot 918.$$

The correlation, therefore, between the size factor of the star and the size of the area of obscuration must lie between 0·9 and 1·0, *i.e.* is very large.

Now I am fully aware of the assumptions made in reaching this result; in particular, of the neglect of the correlation between the size and the chemical constitution. But I think we may safely say that a very high degree of correlation must exist between the size of the star and the size of the obscuration,* and that it is this correlation rather than the common reduction of maximum and minimum amounts of light owing to common distance which is the source of the high correlation between maximum luminosity and light-range in this class of short-period variables. The linear

* See remarks on the bearing of this upon the hypothesis of "sun-spot" or better "star-spot" variability on pp. 132 and 150.

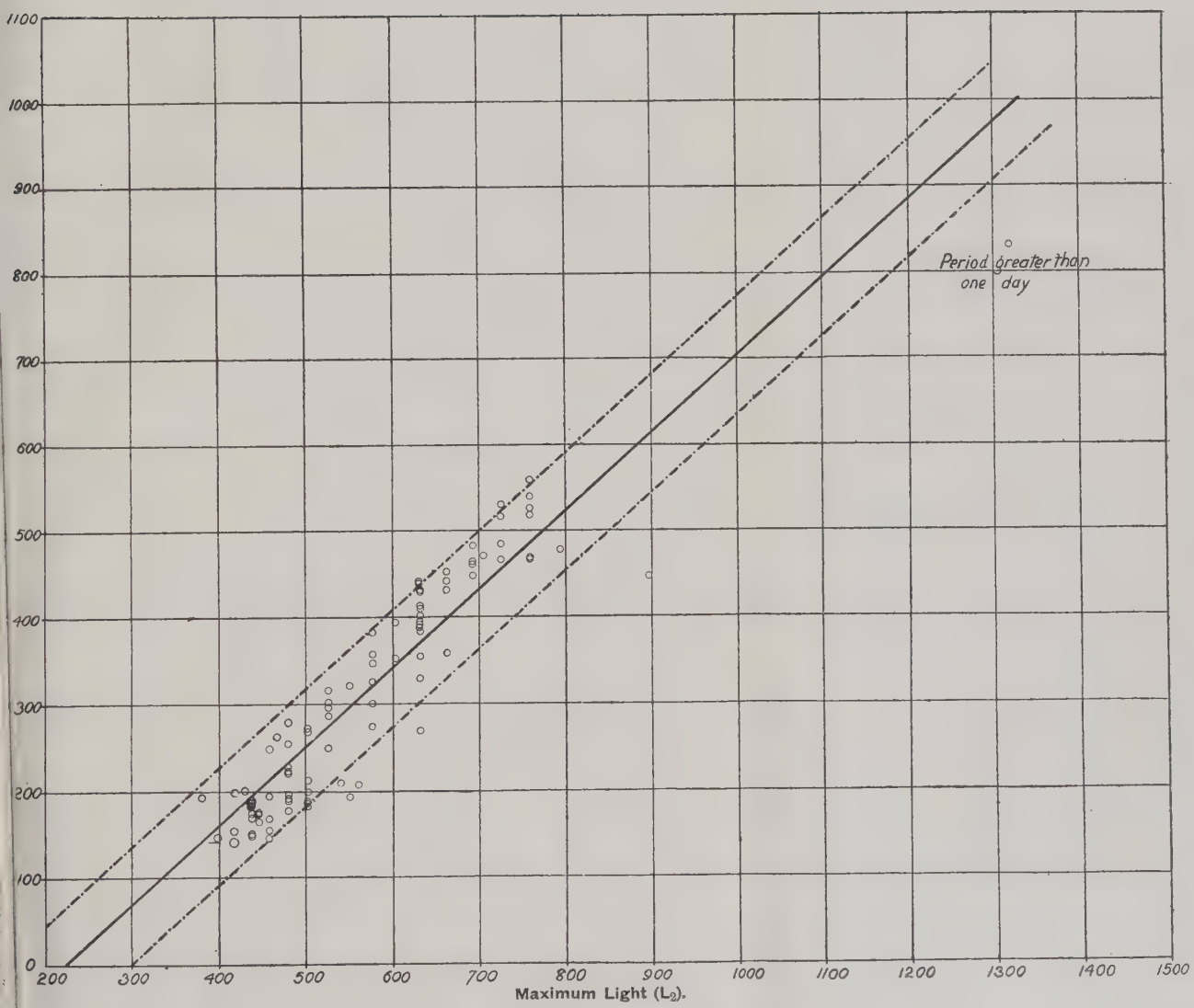


DIAGRAM I.—ω CENTAURI VARIABLES.

Unit of Light = Light of 10th Magnitude Star ('Stellace').

The diagram illustrates how closely these variables actually fall within the limits indicated by the prediction formula.

equation for determining the most probable light-range for a star of maximum light L_2 is

$$L_2 - L_1 = \cdot 906\ L_2 - \cdot 0202 \pm \cdot 0034 \quad . \quad \text{(xxvi.)}$$

The probable error is calculated in the usual way, *i.e.* is equal to $\cdot 6745\ \sigma_{L_2-L_1} \sqrt{1 - r^2_{L_2-L_1, L_2}}$.

It is noteworthy that for $L_2 = \cdot 002$ the light-range becomes zero, *i.e.* for stars of magnitude greater than 16. This is beyond the limits of the series dealt with by Bailey, and its explanation probably lies in the fact that the relationship of light range to maximum luminosity is not truly linear. Diagram I. shows how closely the formula gives Bailey's results. The heavy line is that given by (xxvi). The broken lines represent the range of *twice* the probable error on either side the mean. About 16 stars might be expected to fall outside this belt. There are not quite this number. Those that lie considerably outside it deserve special inquiry and examination.

I think a formula like (xxvi) or the Diagram might be safely used to predict anomalies between maximum and minimum observed luminosities. Of course Equation (i) must first be used to pass from magnitudes to amounts of light. It will readily be recognised how close is the relationship between light range and maximum amount of light ($r_{L_2-L_1, L_2} = \cdot 93$), in stars of this class.

(5) I now turn to the relation between light-fluctuation, *i.e.* $F = (L_2 - L_1)/L_2$ and the period P. We have the following values of the constants:

Mean light-fluctuation	. .	$\bar{F} = \cdot 531$
Variability of light-fluctuation	.	$\sigma_F = \cdot 1225$
Coefficient of variation	. .	$V_F = \cdot 2307$
Mean period	. . . .	$\bar{P} = \cdot 557$ days
Variability in period	. . .	$\sigma_P = \cdot 1685$,,
Coefficient of variation	. .	$V_P = \cdot 3025$,,

Correlation of light-fluctuation and period:

$$r_{F,P} = \cdot 4517 \pm \cdot 0631$$

Probable value of fluctuation for a given period:

$$F = \cdot 348 + \cdot 328\ P \pm \cdot 074 \quad . \quad . \quad . \quad \text{(xxvii)}$$

It will be seen that the correlation $\cdot 45 \pm \cdot 06$ shows a substantial relationship between the period and the light-fluctuation of this class of short-period variables. The degree, however, of association is not nearly so marked as in the case of light range and maximum light, and Equation (xxvii), as seen at once from its probable error, does not enable us to predict as closely as (xxvi). It may still, however, be useful in indicating cases where it is well to test extreme divergence between observations and statistical inference by fresh examination. When I informed Professor Turner of the relation I had found (to be considered below) between light-fluctuation and period in stars of the Algol type, he referred me to

Bailey's paper for somewhat similar material, but he did not confine himself to the reference; he most kindly forwarded with it a correlation between $m_1 - m_2$ and P, which he had worked out to test my result. He found that

$$r_{m_1-m_2,\,P} = \cdot 42.$$

He left out one star with period over a day which I have included. It will be seen that $F = 1 - L_1/L_2$, and $m_1 - m_2 = 2\cdot 5 \log (L_2/L_1) = -2\cdot 5 \log (1 - F)$ are practically equally correlated with P. This, of course, would be theoretically true if the variation in F were small as compared with its mean value, or, what is the same thing, if the coefficient of variation in F, *i.e.* ·23, were small as compared to unity. This it can hardly be held to be, but the close accordance of values shows within what wide limits we reach a practical agreement between the values obtained by correlating first a variable with a second variable and then with any function of this second variable. When we come, however, to deal with such problems as those of the preceding section, we see how impossible it is to work directly with magnitudes.

(6) *Algol Type Short-Period Variables.*—I next turn to the discussion of the short-period variables of the Algol type. Table II. gives the period and light-fluctuation of 57 variables of this tpye. We find—

Including all Stars (57):

Mean Period $\bar{P} = 3\cdot 330$ days, Mean Fluctuation $\bar{F} = \cdot 745$

Variability in period $\sigma_P = 2\cdot 733$ days, Variability in fluctuation $\sigma_F = \cdot 174$

$V_P = \cdot 821$, $V_F = \cdot 233$

$$r_{P,\,F} = \cdot 4109 \pm \cdot 0743$$

Probable value of $F = \cdot 026P + \cdot 658 \pm \cdot 107$

Excluding all Stars with period greater than 4 days (42):

$\bar{P} = 2\cdot 036$ days $\bar{F} = \cdot 675$

$\sigma_P = 1\cdot 092$ days $\sigma_F = \cdot 191$

$V_P = \cdot 536$ $V_F = \cdot 282$

$$r_{P,\,F} = \cdot 5362 \pm \cdot 0741$$

Probable value of $F = \cdot 094P + \cdot 484 \pm \cdot 108$

Retaining only the 15 stars with period over 4 days:

$\bar{P} = 6\cdot 957$ days $\bar{F} = \cdot 787$

$\sigma_P = 2\cdot 684$ days $\sigma_F = \cdot 113$

$V_P = \cdot 386$ $V_F = \cdot 144$

$$r_{P,\,F} = \cdot 1460 \pm \cdot 1723$$

Probable value of $F = \cdot 006P + \cdot 744 \pm \cdot 076$

Now it will be obvious from these results that the positive correlation between period and light fluctuation is confirmed also for these short-period variables.

TABLE II.

	Period.	Light Fluctuation.	Mean of Arrays.
	d		
W Urs. Maj.	0·17	·52	
S Antliæ	·32	·43	
Z_2 Draconis	·55	·48	
T_6 Cygni	·58	·56	
Z_6 Cygni	·63	·60	
U Ophiuchi	·84	·48	d
T_2 Persei	·85	·84	0·71
X_2 Herculis	·89	·43	·52
U_2 Monoc.	·90	·48	
S_3 Carinæ	·90	·64	
Z_2 Centauri	·94	·31	
U Scuti	·95	·52	
R Canis Maj.	1·14	·52	
Z_2 Cassiop.	1·20	·70	
Z Draconis	1·36	·98	
V Puppis	1·45	·48	
U_5 Cygni	1·48	·75	d
U_3 Centauri	1·66	·60	1·59
R_2 Velorum	1·85	·56	·66
X_2 Draconis	1·89	·37	
W_2 Monoc.	1·91	·79	
V_2 Persei	1·97	·90	
W_3 Sagittarii	2·08	·70	
S Libræ	2·33	·67	
S_2 Sagittarii	2·41	·48	
R_3 Centauri	2·48	·77	d
T Cephei	2·49	·86	2·57
W_2 Tauri	2·77	·97	·66
R_2 Draconis	2·83	·96	
W_2 Gemini	2·87	·75	
Algol	2·87	·67	

	Period.	Light Fluctuation.	Mean of Arrays.
	d		
Z Persei	3·06	·96	
R_3 Carinæ	3·30	·43	
Y Camelop.	3·31	·90	
Z_5 Cygni	3·32	·95	
U Sagittæ	3·38	·90	d
Z_4 Cygni	3·45	·90	3·45
U Coronæ	3·45	·73	·78
V Serpentis	3·45	·60	
V_2 Lyræ	3·60	·81	
V_2 Ophiuchi	3·69	·88	
λ Tauri	3·95	·52	
R Aræ	4·42	·64	
V_3 Cygni	4·57	·94	d
R_2 Delphini	4·60	·84	4·60
W Delphini	4·81	·90	·83
V_3 Centauri	5·22	·91	d
T_3 Centauri	5·35	·56	5·50
S Velorum	5·93	·75	·74
X_3 Cygni	6·01	·86	d
R_2 Puppis	6·43	·67	6·48
X_2 Centauri	6·63	·60	·73
Y_2 Persei	6·86	·79	d
V_5 Cygni	8·43	·84	8·96
S Cancri	9·48	·77	·81
S_2 Cephei	12·42	·90	d
W_2 Persei	13·20	·84	12·81
			·87

Taking the period to be under 4 days, we find

$$r_{P,F} = \cdot 54 \pm \cdot 07,$$

and for the ω *Centauri* stars with periods of one day or under we found:

$$r_{P,F} = \cdot 45 \pm \cdot 06$$

the difference $= \cdot 09 \pm \cdot 09$, and accordingly is not definitely significant.

We notice, however, that the correlation of the stars is greatest for those with a period under 4 days, and falls (relative to its probable error) to an insignificant amount for the stars above 4 days period. This decrease in the correlation was first noticed by determining the correlation ratio $\eta_{P,F}$ for the series, and noting how its value, ·57, considerably exceeded that of the correlation coefficient, $r_{P,F} = \cdot 41$. It was thus manifest that the regression line is really curved. This conclusion is well illustrated by the successive slopes of the regression lines; thus we have

Slope for periods under 1 day . . .	·33
Slope for periods 1 to 4 days	·09
Slope for periods 4 to 13 days . . .	·01

The curve, therefore, bears some resemblance to a logarithmic curve, rising at a finite angle at some very small period, and tending to become horizontal as the period increases.*

It may be of interest to note that the correlation of the difference of maximum and minimum magnitudes with the period was worked out for these 57 stars. It was only ·26, showing that the use of magnitude differences may not in all cases be satisfactory as a substitute for the light-fluctuation, *i.e.* we cannot always replace $F = 1 - L_1/L_2$ by $\log (L_2/L_1)$ with profit.

I think we may assert that the Algol type stars as a whole confirm the result of the more homogeneous ω *Centauri* group, *i.e.* that in the case of stars of both types the period is positively correlated with light-fluctuation, but the Algol type stars provide us with the additional information that this relationship is likely to fall when we take stars of increased period.

(8) We have next to inquire how the light range for this series of stars is related to the maximum light. We have already noted the presence of three exceptionally brilliant stars, *Algol*, λ *Tauri*, and V *Puppis* in the series; the constants are worked out with and without this element of heterogeneity. The unit of light is again the stellace.

* Since writing this memoir I have been referred to a paper by S. Beljawsky, *Astronomische Nachrichten*, No. 4238. He gives a graph of the mean values of the amplitude for different periods, and shows that it rises steeply and then becomes almost horizontal. His "amplitude" is proportional to $\log (L_2/L_1)$.

57 *Stars.*	
an light at maximum	$\bar{L}_2 = 40{\cdot}56$
riability of maximum light	$\sigma_{L_2} = 168{\cdot}16$
an light at minimum	$\bar{L}_1 = 15{\cdot}54$
riability of minimum light	$\sigma_{L_1} = 60{\cdot}08$
an range of light	$\bar{L}_2 - \bar{L}_1 = 25{\cdot}02$
iability in range	$\sigma_{L_2-L_1} = 109{\cdot}33$

coefficients of variability are :

$V_{L_2} = 4{\cdot}15$, $V_{L_1} = 3{\cdot}87$, $V_{L_2-L_1} = 4{\cdot}37$

relations :

$r_{L_1, L_2} = {\cdot}9866 \pm {\cdot}0024$,

$r_{L_2-L_1, L_2} = {\cdot}9959 \pm {\cdot}0007$

bable value of range

$L_2 - L_1 = {\cdot}6475\ L_2 - 1{\cdot}2423 \pm 6{\cdot}6736$

54 *Stars.*	
Mean light at maximum	$\bar{L}_2 = 8{\cdot}22$
Variability of maximum light	$\sigma_{L_2} = 15{\cdot}79$
Mean light at minimum	$\bar{L}_1 = 2{\cdot}91$
Variability of minimum light	$\sigma_{L_1} = 6{\cdot}15$
Mean range of light	$\bar{L}_2 - \bar{L}_1 = 5{\cdot}31$
Variability in range	$\sigma_{L_2-L_1} = 10{\cdot}14$

The coefficients of variability are :

$V_{L_2} = 1{\cdot}92$, $V_{L_1} = 2{\cdot}11$, $V_{L_2-L_1} = 1{\cdot}91$

$r_{L_1, L_2} = {\cdot}9499 \pm {\cdot}0090$,

$r_{L_2-L_1, L_2} = {\cdot}9837 \pm {\cdot}0030$

Probable value of range

$L_2 - L_1 = {\cdot}6315\ L_2 + {\cdot}1191 \pm 1{\cdot}2297$

Having regard to the probable errors of the last results, we see that while the high value of the correlation between range of light and maximum light is still maintained, or rather increased, the great variability in light range almost certainly indicates some source of heterogeneity as compared with the group of ω *Centauri* stars.

For prediction we cannot for these variables do better than say :

Probable light range $= {\cdot}65\ L_2 \pm 7$ if we include the three stars,

$= {\cdot}63\ L_2 \pm 1$ if we exclude them.

Thus we see that the correlation of range and maximum light is intensified, but the slope of the curve connecting the two is reduced, *i.e.* the range does not increase so rapidly with the maximum light as for the shortest-period variables.* We have again, indeed, a curved regression line.

(7) *Source of Relation between Period and Light-Fluctuation for Algol Type Variables.*—It may be thought by some that this correlation of period with light-fluctuation may meet with a ready explanation in the case of the Algol variables. I am not convinced, however, that this explanation is valid, and it would not apply to the cluster short-period variables.

Let M_2 be the mass of the bright primary and M_1 the mass of its dark companion. Then the period P will be given by

$$P = 2\pi a^{3/2} / \sqrt{M_2 + M_1},$$

and thus, if the masses of the system were increased, the period would be lessened. Further,

$$F = S_1/S_2.$$

* It might, I think, be convenient to speak of the less than one day variables as "hour variables," those of less than fifteen days as "day variables," and those of longer period as "year variables," expressing their periods respectively in hours, days, and years, or fractions of these units.

Now if we supposed the mass of the dark body to be independent of that of the bright body and gave it a mean or average value, it is clear that P and F would both increase or decrease as the radius of the bright body decreased or increased. In fact, we should have the very correlation between period and light-fluctuation demonstrated above. But a very different result holds if we suppose S_1 and S_2 the size factors correlated. We may investigate this in the following rough approximative manner. S_1 is the occulting area, and not accurately proportional to the square of the radius of the dark body. Still, to a first approximation, we may possibly treat it as on the average proportional. Further, M_1 and M_2 most probably are not of the same density, but, for the purpose of a rough comparison, we will assume them to be so, and give this density its average value for this class of star. Thus we shall write:

$$M_1 \propto S_1^{3/2}, \qquad M_2 \propto S_2^{3/2}$$

$$\frac{\delta M_1}{\bar{M}_1} = \frac{3}{2}\frac{\delta S_1}{\bar{S}_1}, \qquad \frac{\delta M_2}{\bar{M}_2} = \frac{3}{2}\frac{\delta S_2}{\bar{S}_2}$$

Whence

$$\frac{2\delta P}{\bar{P}} = -\frac{\delta M_1 + \delta M_2}{\bar{M}_1 + \bar{M}_2} + \frac{3\delta a}{\bar{a}}$$

$$\frac{\delta P}{\bar{P}} = -\frac{3}{4}\left(\frac{\bar{M}_1}{\bar{M}_1 + \bar{M}_2}\frac{\delta S_1}{\bar{S}_1} + \frac{\bar{M}_2}{\bar{M}_1 + \bar{M}_2}\frac{\delta S_2}{\bar{S}_2}\right) + \frac{3}{2}\frac{\delta a}{\bar{a}}$$

But

$$\frac{\delta F}{\bar{F}} = \frac{\delta S_1}{\bar{S}_1} - \frac{\delta S_2}{\bar{S}_2},$$

or, combining summing and dividing by the number of pairs, we have

$$\begin{aligned} V_P V_F r_{P,F} &= \tfrac{3}{2}(V_{S_1} V_a r_{a,S_1} - V_{S_2} V_a r_{a,S_2}) \\ &\quad - \tfrac{3}{4}\frac{\bar{M}_1}{\bar{M}_1 + \bar{M}_2}(V_{S_1}^2 - V_{S_1} V_{S_2} r_{S_1,S_2}) \\ &\quad - \tfrac{3}{4}\frac{\bar{M}_2}{\bar{M}_1 + \bar{M}_2}\left(V_{S_1} V_{S_2} r_{S_1,S_2} - V_{S_2}^2\right) \\ &= \tfrac{3}{2} V_a (V_{S_1} r_{a,S_1} - V_{S_2} r_{a,S_2}) \\ &\quad - \tfrac{3}{4}\frac{\bar{M}_1}{\bar{M}_1 + \bar{M}_2}(n_1 - n_2) \\ &\quad - \tfrac{3}{4}\frac{\bar{M}_2}{\bar{M}_1 + \bar{M}_2} n_2 \quad . \quad . \quad . \quad . \quad \text{(xxviii)}, \end{aligned}$$

where n_1 and n_2 are the quantities used in Equation (xxiii) above.

Now assume first that there is no correlation between the sizes of the two stars and the size of their orbit. In other words, neglect $r_{a,\,S_1}$ and $r_{a,\,S_2}$. We have

$$\bar{M}_1/\bar{M}_2 = (\bar{S}_1/\bar{S}_2)^{3/2} = \bar{F}^{3/2}$$

and

$$r_{P,\,F} = -\tfrac{3}{4}\frac{1}{V_P V_F}\left\{\frac{\bar{F}^{3/2}}{1+\bar{F}^{3/2}}(n_1-n_2)+\frac{1}{1+\bar{F}^{3/2}}n_2\right\}. \qquad \text{(xxix)}$$

Now in the case of the Algol variables we have from (xvi) and (xviii) for the 57 stars,*

$$V_{S_1}^2 - V_{S_2}^2 = 1{\cdot}8311 = n_1$$
$$V_{S_2}(V_{S_1}r_{S_1S_2} - V_{S_2}) = {\cdot}8536 = n_2$$

Hence by (xxv)

$$r_{S_1,\,S_2} > {\cdot}9962,$$

or, there must be almost perfect correlation between the area of the primary and the area of obscuration. If we assume that the S_1 of (xxv) can be taken even approximately proportional to the area of the dark companion, we can see where (xxix) is likely to lead us. We find

$$r_{P,\,F} = -\tfrac{3}{4}\frac{1}{{\cdot}821\times{\cdot}233}\left(\frac{{\cdot}6430}{1{\cdot}6430}\times{\cdot}9775+\frac{1}{1{\cdot}6430}\times{\cdot}8536\right) = -3{\cdot}5366.$$

In other words, *the correlation would not only be negative but impossible*, i.e. >1. It is thus inconceivable that $r_{a,\,S_1}$ should be zero, or highly probable that the size of the orbit is substantially correlated with the size of the dark companion. We may say, therefore, that the observed correlation of period and light fluctuation cannot be due simply to a relation between the masses of the components and their sizes producing a relation between period and light. Examining Equation (xxviii) we see that the relation cannot be due to $r_{a,\,S_2}$, for this contributes a negative term to $r_{P,\,F}$ which has to be positive. In other words, we want a high correlation between the size of the orbit and the size of the dark

* For the 54 stars: $V_{S_1}^2 - V_{S_2}^2 = -{\cdot}0433$, $V_{S_2}(V_{S_1}r_{S_1,\,S_2} - V_{S_2}) = -{\cdot}0815$. These give no limit to $r_{S_1,\,S_2}$, but for any reasonable values of V_{S_2} and V_{S_1} give high values for $r_{S_1,\,S_2}$. Thus the slender material on p. 144 gives $V_{M_1} = {\cdot}77$ and $V_{M_2} = 1{\cdot}15$, and if we may draw any inference from masses to sizes we should expect V_{S_1} and V_{S_2} not to differ widely from each other or from unity. The following table gives an idea of the values:

V_{S_2}	·75	1·00	1·25	1·5	1·75	2·00	2·25	2·50	2·75	3·00
V_{S_1}	·721	·978	1·233	1·486	1·738	1·989	2·240	2·491	2·742	2·993
$r_{S_1,\,S_2}$	·890	·939	·961	·973	·980	·985	·988	·990	·992	·993

We may conclude, apart from any algebraic limit to the value of $r_{S_1,\,S_2}$, that its range is extremely limited for probable values of V_{S_2} and V_{S_1}. This is further evidence, if any be needed, of the high correlation which must exist between the sizes of double stars of the Algol type.

companion, and this correlation ought to be much more important than that of the sizes of the orbit and of the bright body. The link accordingly between period and light-fluctuation is thus probably indirect. It lies in the following stages: A long period means a big orbit; a big orbit means a large companion; a large companion denotes a large light-fluctuation if an eclipse takes place.

It is possible, of course, that this is only a result of selection. Long periods depending on large orbits, only those large orbits with big dark bodies would have a chance of giving an occultation at all. On the other hand, it may not be selection at all, but a real physical law, whereby in the case of double stars, in particular those of short period, the size of the companion is intimately related to that of the orbit.

If we endeavour to answer this question from the few visible binaries of which we know approximately the masses and the size of orbit, the answer can only be of the vaguest kind, and then it will not of necessity apply to short-period Algol type variables. Still the attempt is worth making, if only for the sake of suggestion. Taking M_2 to represent the larger mass, M_1 the smaller, we have the following table, the unit of mass being the Sun's mass and the unit of distance being distance of Earth from Sun.

TABLE III.

Star.	Distance.	Rank.	M_2.	Rank.	M_1.	Rank.
η Cassiopeiæ	53·3	8	2·90	8	1·10	7
o_2 Eridani	38	7	·82	1	·82	3
Sirius	20	4	2·12	5	1·06	6
Castor	115	10	6·33	9	6·33	10
Procyon	16	2	2·17	6	·43	1
γ Virginis	54	9	1·65	4	1·65	8
α Centauri	23·6	5	1·00	3	1·00	5
70 Ophiuchi	28	6	2·24	7	·56	2
δ Equulei	4	1	·95	2	·95	4
85 Pegasi	16·6	3	7·06	10	4·24	9

The quantities are too vaguely known to admit in this case of any process more suitable than the correlation of ranks.* We have by the sum of mean square difference of ranks for the rank correlations,

$$\rho_{a,M_2} = \cdot 1879, \qquad \rho_{a,M_1} = \cdot 4909.$$

Hence by the formula $r = 2 \sin \rho \frac{\pi}{6}$,

$$r_{a,M_2} = \cdot 20, \qquad r_{a,M_1} = \cdot 51.$$

These values, however slenderly based, are suggestive. There

* Pearson, "On New Methods of Determining Correlation," *Drapers' Research Memoirs*, Dulau and Co.

is a significant correlation between mass and size of orbit, and this is much greater for the secondary body than for the primary.*

In order to obtain confirmatory evidence of this, the correlations between magnitudes and size of orbit were worked out in the same way, the ranks for magnitudes being *reversed* to give light ranks: we have

$$r_{a,\,L_2} = -\cdot002, \qquad r_{a,\,L_1} = \cdot47.$$

Thus, whether we take light (proportional to square of radius) or mass (proportional to its cube), we find that the companion has a size substantially correlated with the dimensions of its orbit, and this in a much higher degree than the size of the primary is correlated with the dimensions of the orbit. Indirectly, perhaps, this may be considered as some evidence of a real physical distinction in origin between primary and secondary.

Thus from rough evidence of the above kind, we see that the first portion of our expression for $r_{P,F}$ (xxviii) must take a substantial positive value.

It is, of course, idle to dogmatise from the results of a few visible binary stars of many years' period to the Algol variables of a few days' or even hours' period. But nevertheless it may not be without suggestiveness to follow up numerically our results. We will assume, on the basis of the slender material discussed above, that

$$r_{a,\,S_1} = \cdot5 \text{ and } r_{a,\,S_2} = \cdot2.$$

We can then evaluate all the terms in (xxviii), instead of only those involving F as in (xxix). We determine, selecting the limit $\cdot996$ for the value of $r_{S_1,\,S_2}$; $V_{S_2} = 2\cdot6496$, $V_{S_1} = 2\cdot9751$, whence

$$r_{P,F} = \frac{3}{2}\frac{V_a}{\cdot321 \times \cdot233}(2\cdot9751 \times \cdot5 - 2\cdot6496 \times \cdot2) - 3\cdot5366$$
$$= 7\cdot5091\ V_a - 3\cdot5366,$$

the last term having the value already found.

Now if we give $r_{P,F}$ the value deduced for the 57 stars, *i.e.* $\cdot41$, we find $V_a = \cdot53$.

If we take all stars with period greater than four days out of the data, we have $r_{P,F} = \cdot54$ and find $V_a = \cdot54$.

This coefficient of variation of the axes of the orbits of the Algol binaries seems very reasonable. The value found for V_a for our slender material of 10 visible binaries is $\cdot8$, and this variability would probably be reduced if we selected out of the binaries such a

* For the few stars in Table III., the correlations found by the method of ranks between the masses M_1, M_2, and the period P, are: $r_{M_1,\,M_2} = \cdot52$, $r_{M_2,\,P} = \cdot20$, $r_{M_1,\,P} = \cdot37$. Also $r_{a,\,P} = \cdot98$. These numbers are of little absolute value, but nevertheless suggestive. They indicate that the partial correlation of period and mass of primary for constant mass of secondary would be practically zero, or the correlation of period and mass of primary may be solely due to the correlation of the two masses, and the fact that the mass of the secondary is correlated with the period.

small group as the Algols. The result suffices to show that the assumption that r_{a_1, s_1} is the factor which produces the correlation between period and light-fluctuation is a reasonable one. At any rate no other solution appears obvious to me at present. Indeed, considering the very large size of the probable error of this ·8, V_a might easily be ·53 for visible binaries. I conclude, accordingly, that there is no inconsistency in the following explanation of the correlation between light-fluctuation and period in short-period variables of the Algol type:

(*a*) There is a high correlation between the size of the components.

(*b*) This causes the mass of the system to have little or no influence on the correlation of light-fluctuation and period.

(*c*) The period, however, is highly correlated with the size of the orbit, and the size of the orbit is very sensibly correlated with the size of the secondary.

(*d*) This correlation of the size of the secondary with the size of the orbit is not wholly or even mainly a result of selection, *i.e.* large secondaries being needful for any occultation at all in large orbits, but is found to hold for the secondaries of visible binary stars, where there is no question of occultation, if we may judge from the small series of which the masses are known. Our conclusions, therefore, from the Algol stars are:

(i) That there is close correlation between the maximum light and the light range.

(ii) That there is considerable correlation between the period and the light-fluctuation.

(iii) That the sizes of the components are closely correlated, having an average ratio of about 5 to 3.

(iv) That the size of the dark body bears considerable relationship to the radius of the orbit—more relationship than that of the primary does.

(v) That the variation V_a of the radius of orbit in short-period variables is probably small. If $V_a = ·5$, say, and we take the mean a to be 3,000,000 miles,* we should have $\sigma_a =$ 1,500,000, or a total range of radius from very small values up to 7,500,000, say.

One or two further points may be noted with regard to these 57 stars. The correlation of maximum and minimum magnitude was ·92. This is, as we might suppose, high because r_{L_1, L_2} is high. On the other hand, $r_{m_1 - m_2, m_2}$ is only equal to ·2. Actually this coefficient has no physical meaning, but only a formal resemblance to $r_{L_2 - L_1, L_2}$; it is really the correlation of log L_2/L_1 with log L_2 and this has no direct physical bearing. The value of $r_{L_2 - L_1, L_2}$ was also found indirectly from the correlation r_{m_1, m_2} of the magnitudes by Equation (ix) and gave the value ·955 as against the direct value ·996. The slight difference, due to the fact that Equation (ix) is only an

* Rough value of Algol's radius merely taken as illustrative.

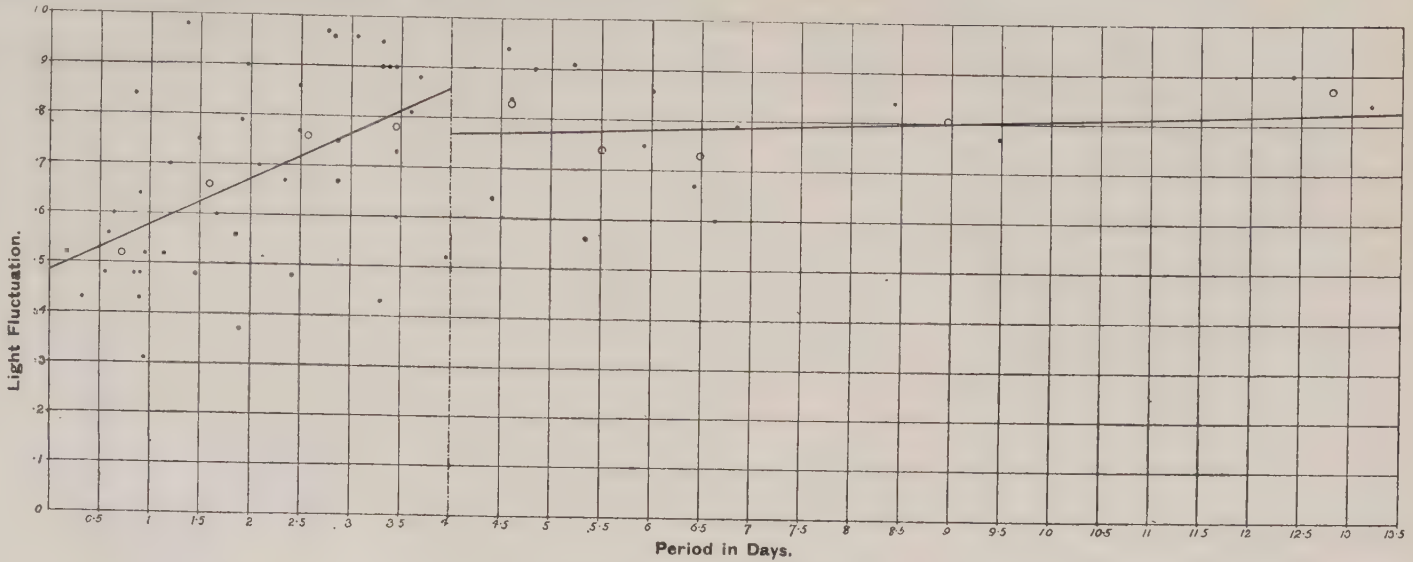


DIAGRAM II.—ALGOL TYPE VARIABLES.

The 57 stars of Table II. are placed on the diagram and the best fitting straight lines to the means (represented by small circles) are drawn (i.) for periods under 4 days, and (ii.) for periods over 4 days, to indicate how the relationship of fluctuation and period decreases with longer periods.

approximation, indicates that the light-result is really involved in the magnitude result.

The limit we have found to r_{S_1, S_2}, *i.e.* $> \cdot 996$ and the value $\cdot 996$ for $r_{L_2 - L_1, L_2}$ seem to indicate that the light-range owes its association with maximum light in the first place to the relationship between the sizes of the primary and its dark companion.

Thus in the Algol type, as in the cluster type short-period variables, there is a sensible relation between period and light-fluctuation, a very close relation between maximum light and light-range, and again between the area of the star and the obscuration.

(8) I now turn to the long-period variables in Chandler's Third Catalogue, referring the reader to my remarks on pp. 133 and 149 to indicate the manner in which the list of stars in Table IV. has been formed. In the following details I give the value of the constants with and without the doubtful 8 stars.* I will take first the question of light-fluctuation:

	87 Stars.	79 Stars.
Mean period	$\bar{P} = 294 \cdot 74$ days,	$\bar{P} = 293 \cdot 84$ days,
Mean fluctuation	$\bar{F} = \cdot 938$,	$\bar{F} = \cdot 968$,
	$\sigma_P = 90 \cdot 25$ days,	$\sigma_P = 84 \cdot 49$ days,
	$\sigma_F = \cdot 1025$,	$\sigma_F = \cdot 0349$,
	$r_{P, F} = \cdot 049 \pm \cdot 072$,	$r_{P, F} = \cdot 201 \pm \cdot 073$.

Now these results indicate that the fairly considerable relation found between period and light-fluctuation in the case of both types of short-period variables is not maintained for the long-period variables. In the case of the 87 stars (period 147 days and upwards) there is no sensible relationship. The 79 stars, however, show that this insignificance is partially due to the fact that the anomalous eight stars with excessively small fluctuations divide themselves into almost balancing groups of relatively long and relatively short periods. The more homogeneous material tends to show some relationship between period and fluctuation, but it has less than half the intensity found for the case of the short-period variables. Secondly, we turn to the maximum light L_2 and the light-range, $L_2 - L_1$.

We have the stellace being the unit of light † :—

* See especially the effect of these 8 stars in Diagram III.

† I have kept more places of decimals than *appear* necessary, because (*a*) all four figures have to be shown in $r_{L_2 - L_1, L_2}$ for the 79 stars if the probable error is to be indicated at all, and (*b*) an erroneous idea on the 'folly,' not to say error, of preserving decimals is current. The value of a constant is its value for the given series, and its probable error is the error of that value. No figures ought to be dropped which can affect what is ultimately a significant figure in a derived constant, and the idea that it is foolish and erroneous to keep figures beyond the probable errors of the earlier calculated constants soon disproves itself if final constants are worked out in different ways. The superfluous figures should only be dropped in reasoning about results. They are preserved here for further calculation on the V's. For example, had we dropped in the 79 stars initially to the limit of the probable error, we should have found $r_{L_1, L_2} = \cdot 41 \pm \cdot 06$, which is substantially different in its inferences from the true value $\cdot 45 \pm \cdot 06$.

	87 Stars.	79 Stars.
Mean maximum light	$\bar{L}_2 = 13{\cdot}325$	$\bar{L}_2 = 12{\cdot}979$
Variability of maximum light	$\sigma_{L_2} = 15{\cdot}756$	$\sigma_{L_2} = 15{\cdot}582$
Coefficient of variation	$V_{L_2} = 1{\cdot}182$	$V_{L_2} = 1{\cdot}201$
Mean minimum light	$\bar{L}_1 = {\cdot}979$	$\bar{L}_1 = {\cdot}344$
Variability of minimum light	$\sigma_{L_1} = 3{\cdot}446$	$\sigma_{L_1} = {\cdot}515$
Coefficient of variation	$V_{L_1} = 3{\cdot}520$	$V_{L_1} = 1{\cdot}495$
Mean range of light	$\bar{L}_2 - \bar{L}_1 = 12{\cdot}346$	$\bar{L}_2 - \bar{L}_1 = 12{\cdot}635$
Variability in light range	$\sigma_{L_2-L_1} = 14{\cdot}870$	$\sigma_{L_2-L_1} = 15{\cdot}357$
Coefficient of variation	$V_{L_2-L_1} = 1{\cdot}204$	$V_{L_2-L_1} = 1{\cdot}215$

Correlations:

$r_{L_1, L_2} = {\cdot}3594 \pm {\cdot}0630$	$r_{L_1, L_2} = {\cdot}4506 \pm {\cdot}0605$
$r_{L_2-L_1, L_2} = {\cdot}9763 \pm {\cdot}0034$	$r_{L_2-L_1, L_2} = {\cdot}9996 \pm {\cdot}0001$

Now we see that while the light-fluctuation, $\bar{F} = {\cdot}94$ to ·97 is essentially different in the case of these long-period variables from the value $\bar{F} = {\cdot}53$ to ·68 of the short-period variables, yet the relation between the light-range and the maximum light shows the same high value. Indeed, to judge by the Bailey stars, the Algol type and these long-period stars, the relationship appears to be intensified as we pass from 'hour' to 'day' to 'year' stars. Thus, taking the homogeneous groups, we have

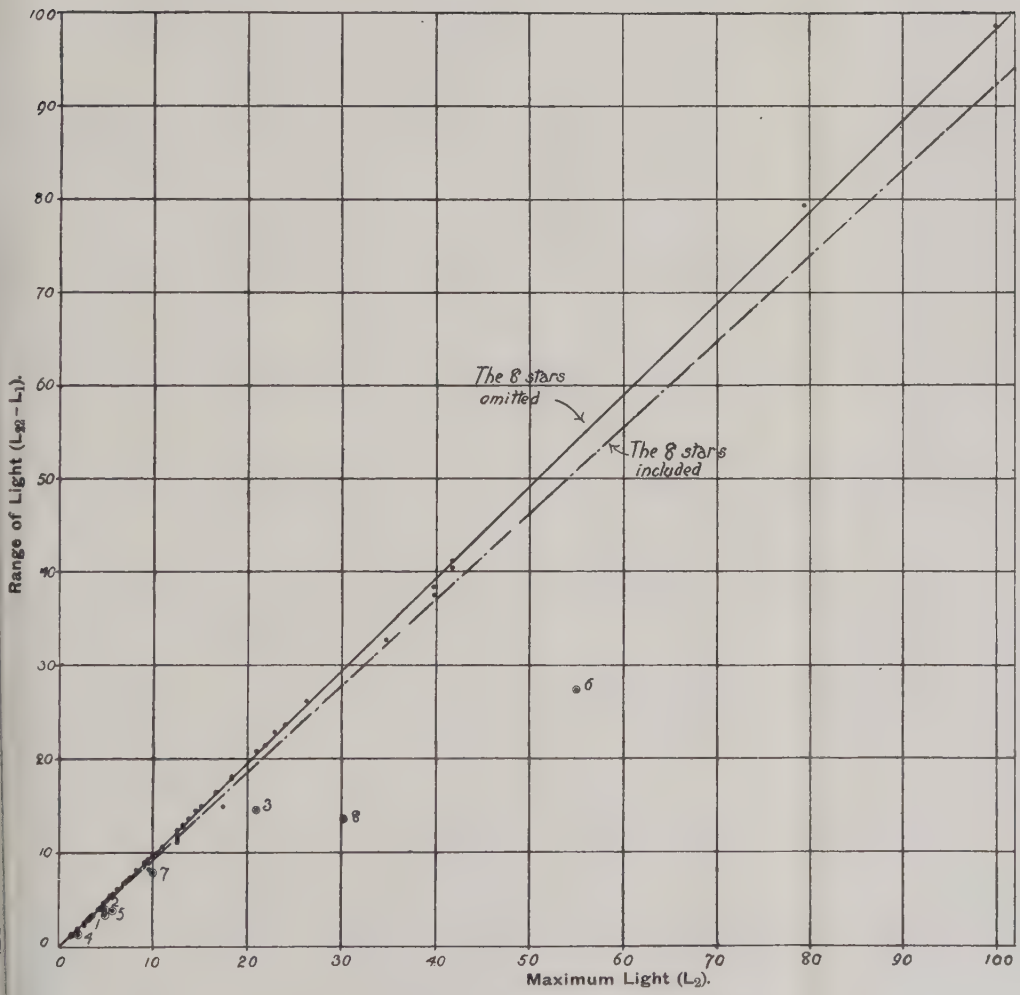
	No.	Mean Period.	Light-Fluctuation.	$r_{L_2-L_1, L_2}$.
ω Centauri 'hour' variables,	91	·56 days	·53	·93
Algol type 'day' variables,	54	2·04 ,,	·68	·98
Long-period 'year' variables,	79	293·84 ,,	·97	·9996

Diagram III. illustrates the closeness with which the 79 stars fall on the regression line, and indicates the anomalous character of the 8 stars noted in Table IV.

On the other hand, we cannot differentiate the long and short-period variables by the fact that r_{L_1, L_2} is small relatively to $r_{L_2-L_1, L_1}$ for them, and not so for the short-period variables. For we have

	Mean Period.	Maximum Light.	r_{L_1, L_2}.
ω Centauri 'hour' Variables	·56 days	·06	·25
Algol Type 'day' Variables	2·04 ,,	8·22	·95
Long-Period 'year' Variables	293·84 ,,	12·98	·45

If the correlation between L_1 and L_2 were merely the result of common distance between the two components we should have expected a greater equality in the values of r_{L_1, L_2} for the different classes of variables, and the reason for the drop in the correlation between the 'day' and 'year' variables would especially need explanation. The low value in the case of the Bailey group may in part possibly be due to uniformity of distance. If an explanation be attempted on this ground, however, it will at once be manifest that common distance will not suffice to explain the remarkably



Anomalous Stars.

1 =	Chandler	976
2 =	,,	3186
3 =	,,	4940
4 =	,,	5156
5 =	,,	7450
6 =	,,	7754
7 =	,,	7783
8 =	,,	8591

DIAGRAM III.—LONG-PERIOD VARIABLES.

The 87 stars of Chandler's Third Catalogue are plotted. They are so largely superposed that it is difficult to exhibit them on this scale. The eight stars which fall badly off the regression line are indicated by enclosing circles. The closeness of the fit is remarkable.

TABLE IV.

Stars taken from Chandler's Third Catalogue.

Index No.	Period in Days.	Light-Fluctuation.	Index No.	Period in Days.	Light-Fluctuation.	Index No.	Period in Days.	Light-Fluctuation.
107	445	·975	4511	257	·993	6608	316	·977
114	320	·984	4557	226	·956	6682	335	·855
466	173	·990	4596	207	·986	6849	351	·984
782	187	·975	4847	376	·994	6905	269	·990
845	167	·993	*4940	384	·698	6921	231	·988
893	236	·990	4948	338	·990	7077	243	·984
906	268	·992	5095	161	·937	7085	191	·952
*976	313	·669	*5156	122	·603	7120	406	·999
1222	210	·987	5157	268	·987	7192	265	·975
1386	253	·970	5190	269	·987	7242	147	·842
1577	325	·987	5194	256	·900	7261	286	·967
1761	380	·956	5237	223	·990	7299	464	·909
1855	460	·994	5338	177	·948	7404	139	·975
1894	218	·967	5504	361	·992	7428	418	·992
1944	413	·939	5511	221	·989	7431	278	·868
1981	313	·917	5583	164	·981	*7450	240	·669
2013	407	·956	5601	328	·977	7468	203	·990
2539	338	·868	5675	357	·967	7560	137	·990
2625	276	·981	5677	357	·997	7609	387	·964
2684	330	·975	5704	277	·993	7659	269	·981
2735	410	·977	5887	303	·931	*7754	132	·522
*3186	482	·780	5889	409	·988	7779	484	·964
3418	310	·988	5950	280	·986	*7783	396	·791
3425	296	·956	5955	246	·990	7999	311	·987
3477	371	·996	6044	308	·996	8040	400	·991
3493	313	·967	6062	282	·900	8324	229	·990
3637	149	·939	6207	348	·981	*8591	360	·475
3825	302	·995	6449	569	·890	8600	430	·990
4492	219	·964	6512	165	·964	8622	350	·963

* Stars marked with an asterisk are those the light-variation of which seems to be markedly different in origin from the bulk of stars in this list. There are four stars in Chandler's list, 806, 2213, 2583 and 4826, with maximum light exceeding 150 of my units, and these have been excluded. My data cover cases from $L_2 = 1$ to 100 stellaces only.

high values found in the same group as well as in the others for $r_{L_1-L_2, L_2}$. It is clear that while something peculiar to *all* variables may account for this, that same character cannot account for the

variations in r_{L_1, L_2}. Hence, as I have indicated on other grounds for the short-period variables, this remarkably close relation between light-range and light-maximum cannot be attributed, I think, solely to common distance. Attributing different origins to the Algol, the cluster, and the long-period variables, we must bear in mind this peculiarity of them all, even if we have to seek different sources for its explanation.

In the case of the Algol variables I have given some reasons for supposing that the correlation $r_{L_2-L_1, L_2}$ is largely a result of close correlation in *size* of the bright and dark bodies. It is worth while inquiring whether we can reach any explanation in a similar manner for the long-period variables. I will suppose the effective luminous area to be reduced, by 'star-spots' or any other process of obscuration, from S_2 to $S_2 - S_1$ at minimum. This may represent a local or a diffused reduction of luminosity; we have no real knowledge at present to enable us to say. But the symbols will allow us at once to apply (xi) to (xiii) without taking S_1 to be the area of any actually occulting dark body or area of obscuration.

We find from (xvi) and (xviii) applied to the long-period variables:

87 Stars.	79 Stars.
$\cdot0525 = V_{S_1}^2 - V_{S_2}^2$	$\cdot0338 = V_{S_1}^2 - V_{S_2}^2$
$-\cdot0065 = V_{S_2}(V_{S_1} r_{S_1, S_2} - V_{S_2})$	$\cdot0135 = V_{S_2}(V_{S_1} r_{S_1, S_2} - V_{S_2})$

The first equations, owing to the negative value of n_2, or $-\cdot0065$, give us no limit to the value of r_{S_1, S_2} at all,* the second equations lead us to

$$r_{S_1, S_2} > \cdot979$$

Hence we have again evidence that the area of obscuration must be closely associated with the area of luminosity. For example, suppose the obscuration due to star-spots, we might safely, I think, predict that the area of the spots will be found closely correlated with the diameter of the star. The weight of the argument in all these cases is in favour of relationship between range and maximum being due to actual stellar conditions, and not to distance from the observer.

(9) *Conclusions.*—I will not venture to be dogmatic about a single conclusion or constant given in this memoir. As a statistician, I am painfully conscious of the dangers of small

* It must not be supposed that because we cannot limit r_{S_1, S_2} its value is necessarily zero. On the contrary, r_{S_1, S_2} will probably have to be very near unity to make n_2 so small a negative quantity. In fact V_{S_1} and V_{S_2} are not improbably of the same order. If that order were 1, we should have $r_{S_1, S_2} = \cdot9935$; if it were only ·3, we should still have $r_{S_1, S_2} = \cdot928$, and for V_{S_2} less than V_{S_2} there will be still higher values of r_{S_1, S_2}. Shortly, any reasonable values for V_{S_1} and V_{S_2} will show a high value for r_{S_1, S_2}, although we cannot with such an equation as that for the 87 stars above, and our present knowledge of V_{S_1}, V_{S_2} assign a definite limit to r_{S_1, S_2}. See also the footnote p. 143.

samples. But certain points have, I think, been indicated which are worthy of the attention of astronomers. For instance:

(*a*) The light-fluctuation is a useful constant, markedly different for 'hour,' 'day,' and 'year' variables, and it might possibly serve to indicate certain stars in both short- and long-period variable lists which deserve special inquiry with a view to ascertaining whether their variability is not of a differentiated character.

Chandler's 7754 or 8591 are such stars, and I hold that Bailey's star 52 is almost certainly a third.

(*b*) There is a quite sensible relation between light-fluctuation (*i.e.* range of light ÷ maximum light) and period. This relation is much higher for the short- than for the long-period variables. It is not, however, demonstrated that it is really zero for the 'year' variables. Its source in the short-period Algol variables is probably the relation between size of dark body and radius of the orbit, which is considerable.

(*c*) There is for both long- and short-period variables an intimate relationship of light-range and light-maximum. This relationship is probably not due to the distance being like for the two phenomena (*i.e.* light-maximum and light-minimum), but to a stellar relationship between the size of the light-giving body and the size of the obscuration. For example, a close degree of association in size between the light and dark bodies in the case of the Algol variables, and between the surface of luminosity and the surface of obscuration in the case of long-period variables.

As usual, a statistical investigation of this kind makes the writer, and I hope the reader, thirst for new material, especially in the way of parallax of double stars, so that more accurate determinations of such constants as $r_{a,\,M_1}$, $r_{a,\,M_2}$, $r_{M_1,\,M_2}$ may be rendered available.

For assistance, not only in laborious arithmetic but in other ways, I have to thank Miss Julia Bell, and for help with material and suggestions at many points, my gratitude to Professor Turner is great. My surmise is, that he will soon be in a position to use my tools with the greater effectiveness that belongs to a craftsman's knowledge of the material to be shaped.

The Comet of 1556; *its possible breaking up by an unknown planet into three parts seen in* 1843, 1880, *and* 1882. By George Forbes, F.R.S.

The writer has never attached undue importance to the speculation advanced by him in 1880 about an unknown planet, but the additional support given to that speculation by the facts now stated, and the greater exactness given to the hypothetical position by using these facts for revising the calculations, seem to call for the present paper. Thus only can observers in the southern hemisphere have an opportunity, if they think fit, of searching for the hypothetical planet, of whose existence, within a few degrees of the place now indicated, the evidence herewith adduced leaves hardly any room for doubt.

1. The comet of 1556 was supposed to be identical with the comet of 1264, and its return about the year 1848 was confidently predicted, and especially by Hind (see *M.N.*, vols. vii., viii., x., also "On the expected return of the great comet of 1264 and 1556," London, 1848; and "The comet of 1556," London, 1857). The identity of the comet of 975 with these was also suspected.

Elements of the orbit of comet 1264 have been computed by Dunthorne, Pingré (two), and Hoek (three). These and all other elements of orbits referred to in this communication are given in Dr. J. G. Galle's *Cometenbahnen*, Leipzig 1894.

Elements of the orbit of comet 1556 have been computed by Halley, Hind (three), and Hoek. These are all deduced from the observations of Fabricius, but the later computers had also access to the far better observations of Joachim Heller.

In the following elements of the orbit of comet 1556, as in other citations in this communication ω is what Galle calls the Argument des Perihels and is the angle from ascending node to perihelion measured in the direction of comet's motion.

Elements of Comet 1556.

Perihel. Passage.	ω.	Ω.	i.	Log Perihel. Dist.	Computer.
1556.	° ′ ″	° ′ ″	° ′ ″		
Apr. 21·842	103 8	175 42	32 6 30	9·666424	Halley.
Apr. 21·8077	91 3 54	176 33 48	36 11 24	9·75246	Hind.
Apr. 21·0037	86 20 0	176 29 6	36 39 12	9·78254	*idem.*
Apr. 22·0298	98 49·1	175 25·8	30 12·2	9·70323	*idem.*
Apr. 22·1911	100 52·6	175 13·9	32 25·7	9·69092	Hoek.

[Mean Equinox, 1556·0.]

Now, if the non-appearance of the comet in 1848 as expected was due to a complete change in its orbit by the perturbations of an unknown planet, it is certain that this would be most likely to

happen at or about its aphelion position, which in normal conditions it would reach in the year 1702 A.D.

The longitudes (L) and latitudes (λ) of aphelion and L′, λ' of a point on the orbit 2° in advance of aphelion deduced from the elements above are—

L.	L′.	λ.	λ'.	Computer.	
106 11	108 27	−31 10	−30 52	Halley	M.E. 1908·0.
92 58		−36 11		Hind I	
87 0		−36 34		Hind II	
100 36	102 59	−29 48	−29 37	Hind III	
103 9	105 29	−31 46	−31 31	Hoek	

2. The comets 1843 I., 1880 I., and 1882 II. have been long recognised as a group because of the resemblance in their orbits. These are certainly not identical; but they all have retrograde motion, and they all approach very close to the Sun's surface at perihelion, and the other elements are not very different, as shown below.

puter.	Perihelion Passage.	ω.	Ω.	i.	Log. q.	Excentricity.
bbard	1843, Feb. 27·41702	82 34 38	1 14 55	144 19 21	7·7433765	0·99991572
utz	1880, Jan. 27·62502	86 18 7	6 10 29	144 39 42	7·739478	
utz	1882, Sept. 17·23051	69 34 35	346 0 43	141 59 45	7·8889895	0·9999330

In each case the last orbit quoted by Galle has been adopted as being the best. In the case of comet 1843 I., there are 18 orbits in Galle, and some of the orbits first computed differed sensibly, but the later ones not much, and Hubbard's work is generally recognised as the best. In the case of comet 1880 I., there are 10 orbits in Galle, all agreeing pretty well; Gould's orbit is almost identical with that of Kreutz, and Hind's is very similar. In the case of comet 1882 II., there are 34 orbits in Galle, all agreeing fairly well together.

The point of similarity between the orbits of these three comets to which attention is now to be directed is the following :—

The position of aphelion of each comet is in the same part of the heavens. The following table gives the longitude and latitude of the aphelia, (L and λ), and also of a point further advanced on the orbit by 2° (L′ and λ'), to show the plane of motion.

Comet.	L.	λ.	L′.	λ'.
1843 I.,	101 17	−35 35	98 51	−35 30
1880 I.,	101 6	−35 14	98 39	−35 19
1882 II.,	101 40	−35 13	99 18	−35 44
	Aphelion.		2° in advance of Aphelion.	
Mean	101 21	−35 21	Mean Equinox 1908·0.	

These three aphelion positions are practically identical.

No attention, so far, has been paid to the aphelion distances deduced from any elliptic orbits that have been assigned to these comets. But the identity of the aphelion directions coupled with the variation in other elements leads to the inevitable conclusions that :—

These three comets have elliptic orbits.

They were one body when last in aphelion, and then separated in slightly different orbits.

On comparing the aphelion directions of comet 1556, and recognising the degree of uncertainty attaching to them, the conclusion is almost irresistible that the lost comet 1556, when at aphelion, was disturbed by an unknown planet and broken up into at least three parts.

Acting upon this supposition, it is easy to supply very nearly the one uncertain element of these three comets.

Taking 292 years, from 1264 to 1556, as the period of that comet, its mean distance is 44·01 ; and, from the perihelion distance given by Hoek, the aphelion distance of comet 1556 is 87·52. This may be taken also as very nearly the aphelion distance reached by the three comets 1843 I., 1880 I., 1882 II., at or about the date A.D. 1702.

It may be well here to note the aphelion velocities :—comet 1556, $v = +\cdot07072$; comet 1843 I., $v = -\cdot00753$; comet 1880 I., $v = -\cdot00742$; comet 1882 II., $v = -\cdot00891$.

Since these motions are all nearly parallel to the ecliptic, it may be noted that the velocity to be given by the unknown planet to comet 1556, to transform its orbit into one of the others, is nearly the same in each case, and is less than $-\cdot08 = 0\cdot24$ miles per second. As before, the unit velocity is one Earth's distance from the Sun in one year.

3. An unknown planet, whose existence was suspected, at a distance from the Sun equal to about 100 times the Earth's distance, had a hypothetical orbit and position in that orbit assigned to it by the writer in 1880.* That distance was assigned from the fact that seven comets appeared to have their aphelia at about that distance, and from the analogy of comets whose aphelia are similarly associated with Jupiter and Neptune. An orbit was assigned by noticing that four of these aphelion positions seemed to lie on one plane passing through the Sun ; and a position in that orbit was assigned by comparing the mean motion of a planet at such a distance with the dates when the comets reached their aphelia.

From these data the orbit was defined as follows : "The longitude of its ascending node is at 250°, and it cuts the ecliptic at an angle of 53°." As to the position of the planet in its orbit the following statements were made : "comet IV. was influenced by the planet in the year 409 A.D. ; comet II. in 1655. The interval is 1246 years. The longitude of aphelion of comet IV. [measured on the above orbit] is 29°. That of comet II. is 115°,

* See *The Observatory*, June 1880. Also *Proc. R.S.E.*

which added to 360° gives 475°. The difference is 446°, which is accomplished in 1246 years, which gives 2·794 years to 1°; and a complete revolution in 1006 years, agreeing remarkably with the period determined from the assumed distance of the planet."

Then again: "Returning now to the hypothetical planet, if we wish to determine the present position (A.D. 1880) of the planet. The mean of the four dates of aphelion disturbance by the hypothetical planet is the year 1160 A.D., and the mean longitude of the planet at these four times is 287°. From the year 1160 to 1880 it must have passed over 258°, whence its present position in its orbit (A.D. 1880) is in longitude 185°."

From the above data the hypothetical planet in 1702 would be in longitude 106° 59′ and latitude −38° 33′; and after covering two degrees further in its orbit (in about 1708) its position would be longitude 108° 55′; latitude −39° 49′.

This is quite near the aphelion positions of comets 1556, 1843 I., 1880 I., and 1882 II. Producing backwards the supposed path of the planet, the proxmity is at first startling.

The coincidence loses, however, some of its apparent importance when it is remarked that such proximity was to be expected from the fact that the comet 1843 I. was one of those whose aphelion positions were found to lie on one plane through the Sun at about the same distance. On the other hand, this fact makes the chances of finding the planet greater.

4. By inspection of the accompanying chart, the three coincidences to which attention has been drawn are made very clear.

By far the most important of these is the second. That three distinctly different orbits of comets should all intersect at one point is almost a certain proof that these were at that point broken up from a single comet, either by explosion or by the influence of a planet in close proximity.

The latter explanation is rendered more probable when we remember that Lexell's comet thus twice approached Jupiter so closely as to have its orbit completely changed.

The probability of this explanation is increased by the fact that the point of assumed disturbance was at the comets' aphelia, where the slow motion gives a planet the best chance of disturbing the orbit. It was at aphelion that Lexell's comet was disturbed.

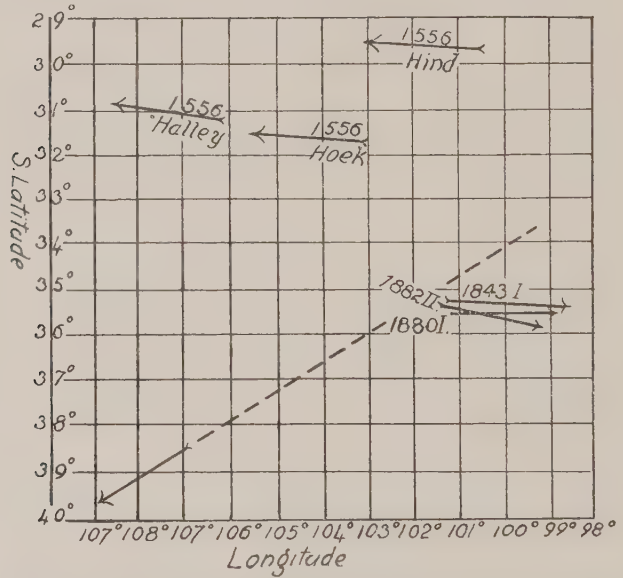
The non-agreement of calculated aphelion distances of these three comets counts for nothing, because all of them had such extremely small perihelion distances. In such a case it is impossible to arrive with any certainty either at the period or at the aphelion distance by calculation from the observations.

The evidence that the comet which was thus broken up was the comet of 1556 is threefold:—(1) the non-appearance of that comet about 1848 as expected; (2) the good agreement of the dates of appearance of all three comets with the supposition of their having been at or about their aphelia in 1702; (3) the agreement of the directions of aphelia in spite of the difficulties there have been in interpreting the observations of 1556. This last coinci-

dence is the more notable from a search made by me through all the cometary orbits recorded by Galle, and, except comet 1887 I., there is no other that falls within many degrees of the aphelia of the three comets of our group.

The change from direct to retrograde motion is no objection. The total negative velocity given by the disturbing planet to comet 1556, to convert the orbit into that of any of the three, is very small.

If this evidence be accepted, it helps most materially in a search for the planet. The identity of comets 1556 and 1264 gives the period, and thence the aphelion distance 87·52, and the date of aphelion 1702. These are the distance and date likewise for the



supposed planet, while the latitude and longitude of the planet at that date are best given by the far more accurate observations of the three 19th century comets, taking the mean of their aphelia.

Starting with the known position (including radius vector) of the unknown planet at a certain date makes the quest far more promising, and this is a far more useful result than anything attempted in the work on the subject in 1880.

That investigation remains, however, the only guide as to the path of the planet since A.D. 1702 and its mean motion.

A new calculation, with fuller data, has therefore been undertaken; and this was necessitated by the fact that while the mean distance of the planet is 100 or more, its distance from the Sun in 1702 was only 82·52. Assuming this to be the planet's perihelion, we must use an orbit the excentricity of which materially affects

the calculations. The work done in this revision will now be given.

5. Before stating the results of a revision of the work in 1880, it must be mentioned that at that time an alternative position of the planet was given on the supposition that its orbit was near the ecliptic.

The reason was that by this means it had been possible to find the true position of Neptune within three degrees by using the five available comets, some of whose aphelia were not near the ecliptic. Four of these were found to have aphelion dates that fitted the possible motion of a planet. Three out of these four aphelion positions were near the ecliptic. This now seems wrong, and the Neptune group ought to have been taken to support the orbit of the unknown planet not on the ecliptic; in fact, the orbit which explains the apparent disappearance of comet 1556.

It must further be stated that those who have searched for the planet have done so only in the ecliptic position.

The writer must here explain that he would certainly have searched the other place, but shortly after he wrote that paper he devoted himself to assisting to lay the foundations of the profession of electrical engineering, and his time was till lately so absorbed by these duties that he was never free to continue his search for the planet.

6. Revision of the work published in 1880.

This work was based on the discovery that there seemed to be seven comets whose aphelion distances from the Sun were about 100 times the distance of the Earth. This seemed to make it certain, from the analogies of the Jupiter and Neptune groups of comets; and from the work of Professor Newton on the capture of comets, which gave an explanation of that connection between comets' aphelia and planetary distances—that there is an unknown planet at or about the mean distance 100, revolving round the Sun in about 1000 years.

Out of these seven aphelia, four were found to lie almost exactly on a plane passing through the Sun. [An error of computing has been now corrected in this part of the work on comet 1855 II.] It seemed likely that this might be the true plane of the planets' orbit.

The dates at which the four comets were in aphelion were found to agree fairly with the dates at which a planet whose period is 1006 years, revolving in a circular orbit, might have been in these four positions. Thus an estimate was made of the actual longitude of the unknown planet in its orbit.

The first step undertaken now has been to compute from every set of elliptic elements of all the comets in the Catalogue of Galle the corresponding aphelion distances, so as to find corroborative evidence.

Some of the comets have very wide ranges in the aphelion distances assigned by different computers of elements. There are only 17 comets in which the range of aphelion distances given by

any computers includes the distances entitling them to be included in this investigation, viz. between 85 and 135 Earth's distances.

Out of these 17 comets a great many deserved but little attention, owing to obvious doubt as to the value of the excentricity. Six comets, however, out of this class are found to lie on or close to the plane of the orbit suspected in 1880. This comes from the addition, to the four already used, of comets 1855 I. and 1882 II. Curiously enough, all the aphelion distances computed for Comet 1880 I. are very small. This is due to uncertainty, owing to the very small perihelion distance combined with the desire of computers to identify this comet with comet 1843 I. Otherwise, as shown in § 2 of this paper, Comet 1880 I. should be added to our list; also comet 1556,—thus raising the number of comets whose aphelia lie on or close to the plane chosen to eight. Four of these, however, refer to the same position and the same date.

The comet 1843 I. was one of those used in 1880. The other three which were then used, as they now are to be used, are 1840 IV., 1855 II., and 1861 I.

Concerning comet 1840 IV., out of eight sets of elements given by Galle three are elliptic, giving aphelion distances 97, 97, and 101. Of these, two are by Goetze. The third of these, the result of a laborious research, is by Schulze.

Concerning comet 1861 I., nine orbits are given by Galle. Three are elliptical—one by Pape, two by Oppolzer—giving aphelion distances 300, 110, 110. Oppolzer's final result gives a period of 415·430 years. This comet approaches the Earth's orbit very nearly (about 0·0022), but, as I explained in 1880, it certainly was seen at three previous revolutions, in 1444, 1032, and 617 I. In each of these returns the Earth's position in its orbit was far away from the comet, and the orbit could not be disturbed. The proofs of identity are as follows.

Comet 1861 I. shone in the beginning of May as a large nebula with a small nucleus and a tail only 3° long. Its greatest brightness was equal to a star of the 2nd. or 3rd. magnitude. I have computed from its orbit that if the Earth had been in the position it occupies in July the comet would have passed a little to the east of β Leonis, going towards α Virginis.

Now, comet A.D. 1444 is thus described by Biot: "On August 6 a comet 10° long was seen to the east of β Leonis. It became longer day by day till August 15, when it entered the sidereal division of α Virginis and disappeared."

Comet A.D. 1032 is thus described by Biot: "On July 15 an extraordinary star appeared in the N.E. It approached β Leonis and threw out a tail."

Comet A.D. 617 I. is thus described by Ma-tusan-lin: "In July a comet, with a tail 3° or 4° long, was seen near β Leonis; after some days it disappeared."

All three comets agree with the dates calculated from Oppolzer's orbit. All were seen about July in the direction to be expected from Oppolzer's orbit. All lasted visible to the naked eye only for

a short time. In two cases the length of tail is given, and was only 3° to 10° long, as in 1861.

I think we may safely put the aphelion date when the planet was at the same place as being A.D. 409, certainly not later.

Concerning comets 1855 I., 1855 II., no remark is required except that there is nothing to indicate any previous appearance of either.

To comet 1855 I., Tiele has ascribed two elliptic orbits, aphelion distances 206 and 124; periods 1058·6 and 520·12 years. Thus the date of aphelion is a little uncertain.

To comet 1855 II., Donati gave an elliptic orbit, aphelion distance 124, period 493 years, subject to some uncertainty.

The following elements of orbits are adopted in this investigation :—

puter.	Perihelion Passage.	ω.	Ω.	i.	Log. q.	e.
		° ′ ″	° ′ ″	° ′ ″		
ıltz } ıheil }	1840, Nov. 13, 67060	133 36 1	248 55 48	57 58 6	0·1705856	0·9711151
e	1855, Feb. 5, 05384	323 5 59	189 43 33	128 35 41	0·3411427	0·965185
ati	1855, May 30, 20781	22 39 5	260 15 7	156 52 52	9 7542137	0·9909006
olzer	1861, June 3, 39641	213 26 19	29 55 42	79 45 31	9·9641181	0·98346314

[Mean Equinox, 1840·0; 1855·0; 1855·0; 1861·0.]

From these the longitudes and latitudes of aphelion are deduced. Those of the other comets are grouped together, using the mean values already determined. The aphelion distances computed by different calculators are also given for each comet.

Comet.	Aphelion Distances.	Aphelion. Longitude.	Aphelion. Latitude.
		° ′	° ′
1840 IV.,	97, 97, 101	220 56	−37 53
1855 I.,	206, 124	35 36	+27 59
1855 II.,	124	60 1	− 8 42
1861 I.,	300, 110, 110	37 18	+32 50
1843 I.,	7, 69, 16, 63, 21, 22, 173, 131		
1880 I.,			
1882 II.,	83, 23, 150 to 200 (18 values), 510, 247, 261, 231	101 21	−35 21
1556.	87·52		

[All these longitudes are reduced for precession to the mean equinox 1908·0.

All of these aphelion positions lie almost on one plane passing through the Sun, the two comets of 1855 showing a slight divergence. The plane was necessarily chosen to pass through the mean aphelion position of the group of three comets. It differs little from the plane chosen in 1880. The longitude of the ascending node is 247° 34′, and the inclination is 52° 0′ 30″.

In the following table the longitude is measured from the first point of Aries along the ecliptic to the ascending node, and thence along the supposed planet's orbit, and the latitude is the distance from the assumed plane of the orbit.

Comet.	Dates of Aphelia.	Orbital Longitude.	Orbital Latitude.
		° ′	° ′
1840 IV.,	A.D. 1659, 1287, 915 (Schulze)	196 6	+ 0 13
1855 I.,	A.D. 1575 (Tiele)	40 17	− 4 34
1855 II.,	A.D. 1609 (Donati)	69 53	−11 18
1861 I.,	A.D. 1654, 1239, 824, 409 (Oppolzer)	24 6	0 0
1843 I., 1880 I., 1882 II. 1556	A.D. 1702	114 57	0 0

It might be thought sufficient to assume a circular orbit for the planet of a radius about 100. But the transformation of the orbit of comet 1556 into three makes it nearly certain that in 1702 the planet was close to the comet, at a distance 87·52. If the mean distance is to be maintained at about 100 the orbit is excentric, and the orbital longitude 114° 57′ should be taken as the most probable longitude of perihelion, and 87·52 as the most probable perihelion distance. It is only by trial and error that the mean distance can be determined.

A cursory examination shows that, as in the paper of 1880, the fundamental feature of a solution lies in assuming that comet 1861 I. was affected by the planet at the fourth aphelion passage, counted backwards from now, in A.D. 409. The date of that aphelion passage can be fixed with certainty, as the comet's four returns have been recorded, and therefore its period is known.

This fixes the mean distance and excentricity of the orbit of the hypothetical planet.

Ω.	*i*.	*e*.	π.	Mean Distance.	Date of Perihelion Passage.
° ′	° ′ ″		° ′		
247 34	52 0 30	0·1665	114 57	105·1	A.D. 1702

The period of revolution is 1076 years.

The next feature, as in 1880, is that comet 1840 IV. must have been affected by the planet at its third aphelion passage counted backwards from now, about the 9th or 10th century. The impossibility of verifying previous appearances of this comet renders its period dependent entirely on the three sets of calculated elliptic elements which give aphelion distances 97, 97, and 101 Earth's distances. A mean value among these gives the date of aphelion A.D. 821, fitting in exactly with elements assigned to the planet, while the planet's radius vector in that longitude would be 99·88.

Comets 1855 I. and 1855 II. must have been affected at their last aphelion passage. Donati gave to comet 1855 II. an aphelion

distance 124. I find that by assuming an aphelion distance 125·75 the comet and planet were both in the longitude of the comet's aphelion in A.D. 1608.

For comet 1855 I. Tiele's two elliptic orbits give aphelion distances 206 and 124. I find that this must be changed to 147·27 to bring the comet to its aphelion at the date A.D. 1532, when the planet reaches the same longitude.

Considering that the assumption made as to perihelion distance and longitude of perihelion of the planet's orbit is only the most probable, it will be recognised that these results go far in support of the hypothesis.

These results are shown in tabular form as follows:—

et.	Aphelion. Long. (Orbital.)	Aphelion. Lat.	Aphelion Distance calculated.	Aphelion Distance assumed.	Date Aphelion.	Date for Planet in same Long.	Rad. Vect. of Planet.
	° ′	° ′					
I. I. II.	114 57	0 0	87,52	87·52	A.D. 1702	A.D. 1702	87·52
I.,	24 6	0 0	110,36	110·36	409	409	102·7
IV.,	196 6	+ 0 13	97;97;101	99·70	821	821	99·88
I.,	40 17	− 4 34	206; 124	147·27	1532	1532	
II.,	69 53	−11 18	124	125·75	1604	1604	

It is worth noting that the aphelion distance assumed for comet 1840 IV. is 99·70; the values from computed orbits being 97, 97 and 101; while the radius vector of the planet at that longitude is 99·88, the orbit having been computed without using this comet. This remarkable agreement makes it nearly certain that the perihelion distance of the planet and its longitude of perihelion, which were chosen as probable, are not far wrong.

It only remains now to fix the present position of the planet. Taking A.D. 1702 as the date when the planet was in longitude 114° 57′, the planet's longitude in its orbit in 1908 is 201° 57′; and its position with respect to the ecliptic is A.D. 1908 longitude 215° 31′; S. latitude 33° 53′; A.D. 1914 longitude 217° 21′; S. latitude 32° 15′.

When searching for the planet it must be remembered that probably the orbit is subject to less uncertainty than the epoch, and the search should be extended forward and backwards on the line indicated by the above positions.

Such a search cannot be prosecuted except at a southern observatory.

7. Perturbations of Neptune's longitude. It ought to be mentioned here that in 1906 the writer computed roughly the perturbations that might be expected on Neptune's longitude and radius vector by the hypothetical planet on any assumption as to the mass of that planet, concerning which mass we can form no opinion.

These calculations were based upon the orbit of a planet assumed to be in the ecliptic, a hypothesis now abandoned. But the results have still a value as showing that up to the present date the presence of such a planet could not be distinctly indicated, but that in the next 30 or 40 years it should become very evident. This arises from the fact that Neptune has been observed since 1846 only, and in 1795 (Lalande's two observations), while its period is 165 years. Consequently it has been possible to combine the perturbed positions, by a suitable choice of excentricity and mean motion, to satisfy an apparently undisturbed orbit (as Bouvard did with observations over a part of the orbit of Uranus). But this treatment cannot be continued over a whole revolution.

In the following table, δL indicates the retardations that would be produced on Neptune's longitude if the mass of the planet equals that of Jupiter. Obviously, if it equals that of Neptune, these perturbations must be divided by 20. $\delta L'$ indicates the retardations that might be produced by using an undisturbed orbit, and changing the excentricity and mean motion by certain selected amounts. The second table shows Newcomb's residuals.

Date.	δL.	$\delta L'$.	(C—O) or (δL—$\delta L'$)
	″	″	″
1795	−14·46	−16·66	+2·20
1846	− 0·16	− 0·14	− 0·02
1855	0·00	+ 0·09	− 0·09
1864	+ 0·16	+ 0·53	− 0·37
1873	+ 1·18	+ 1·64	− 0·46
1883	+ 3·64	+ 3·83	− 0·19
1892	+ 7·47	+ 7·37	+ 0·10
1901	+11·67	+12·35	− 0·68
1910	+14·44	+18·70	− 4·26
1919	+12·98	+26·20	−13·22

	Newcomb 4-year Groups C—O.
	″
1795	+2·05
1848	+0·27
1852	−0·26
1856	−0·07
1860	+0·52
1864	+0·37
1868	−0·12
1872	+0·30
1876	−0·20
1880	−0·32
1884	+0·36
1888	+0·30
1892	−0·25
1896	+0·28

These tables show why we could not expect to detect the presence of such a planet, by observations of Neptune to this date, unless the planet's mass was enormous.

But if the planet exists, its presence, even with moderate mass, must eventually be thus revealed.

Erratum.

In Professor Turner's paper, p. 61, first line of paragraph 2,

for loss of light to the ether
read loss of light due to the ether.